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Two-dimensional piezoelectricity. Part I: eigensolutions of nondegenerate and degenerate materials

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Abstract

General solutions of two-dimensional piezoelectricity, which yield all solutions of 2-D boundary values problems, are obtained by combining four complex conjugate pairs of independent eigensolutions, each containing an arbitrary analytic function. The forms of representation are fundamentally different for 14 different classes of nondegenerate and degenerate piezoelectric materials, as determined by the multiplicity and types of eigenvalues. Degenerate materials possess *high-order eigensolutions*, in which the eigenvectors of equal and lower orders are intrinsically coupled. Such coupling is nonexistent in nondegenerate cases including the well-known and analytically simple case with no multiple eigenvalues. The present analysis is drastically simplified by using the *compliance-based formalism*, instead of the stiffness-based, extended Eshelby–Stroh formalism. Explicit expressions are obtained for the eigensolutions, the pseudometrics, and the intrinsic tensors characterizing piezoelectric materials of every type.

Keywords: Piezoelectricity; Anisotropic elasticity; Degenerate materials; Eigensolutions; Derivative rule

1. Introduction

Piezoelectric materials have received considerable attention because of their use in monitoring and controlling structural components. Such materials show coupling effects between electric fields and elastic response. The constitutive equations of linear piezoelectricity are well known. Based on these equations, a number of three-dimensional and two-dimensional solutions for piezoelectric materials (Sosa, 1991; Pak, 1992; Sosa and Castro, 1994; Suo et al., 1992; Ruan et al., 1999; Shodja and Kamali, 2003), and homogeneous or multilayered piezoelectric plates have been obtained (Ray et al., 1992; Mitchell and Reddy, 1995;

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Bisegna and Maceri, 1996; Huang and Wu, 1996; Heyliger, 1997; Vel and Batra, 1999). Nevertheless, the subject has not been as fully developed, in an analytical sense, as anisotropic elasticity. Various important problems, such as fundamental singular solutions (Green's functions) for some simple domains, problems of infinite regions with holes and inclusions, etc., have been investigated sparsely for nondegenerate piezoelectric materials, let alone for the degenerate cases.

The remarkable richness of results in two-dimensional elasticity suggests that many of such results may be extended to two-dimensional piezoelectricity. This is due primarily to the feasibility of using analytic functions of complex variables to represent the general solution. For *nondegenerate* piezoelectric materials, the complex variable formulation has been developed by extending the Eshelby–Stroh formalism of anisotropic elasticity to include the electric effect (Ting, 1996). This yields a characteristic equation of the eighth degree for four complex conjugate pairs of eigenvalues, and a system of four linear equations for the eigenvectors associated with the three displacement components and one electric potential. However, the eigenvectors of the Eshelby–Stroh formalism have very lengthy analytical expressions in comparison with the eigenvectors of the compliance-based formalism, even if the material is nondegenerate. [For the various degenerate cases in plane elasticity, the simple expressions of the latter formalism (Yin, 2000a) are contrasted to the lengthy expressions of the Eshelby–Stroh formalism (Yin, 2000b)]. But explicit expressions of the eigenvectors are required in various analytical formulations and solutions of piezoelectricity, e.g. boundary integral equations and Green's functions of infinite and finite domains.

It is also well-known in 2-D elasticity that the representation of the general solution of nondegenerate materials, and all particular solutions deduced from it, cannot be applied to the various degenerate cases, including the important case of isotropic and transversely isotropic materials. In such cases, there are high-order eigensolutions associated with a repeated eigenvalue. A kth-order eigensolution is not characterized by a single eigenvector, but shows intrinsic coupling of $k+1$ eigenvectors with orders increasing from 0 to k. This results in complication of analysis (for example, the appearance of higher-order kernel functions in boundary integrals) that is not found in the nondegenerate cases. At the same time, it also causes some simplification of the results and expressions due to a reduced number of distinct complex variables.

That the stiffness-based formalism yields more complicated expressions of the eigenvectors than the compliance-based formalism in the nondegenerate case of 2-D elasticity has been remarked by Stroh (1958). The disadvantage of the former becomes even more pronounced in the various degenerate cases, where one needs to obtain the higher-order eigensolutions by differentiating the zeroth-order eigensolutions repeatedly with respect to the eigenvalue (Yin, 2000a,b). In contrast, the compliance-based formalism reduces the key eigenrelation to a linear equation involving a 2×2 eigenmatrix $\mathbf{M}(\mu)$, whose solution can be obtained effortlessly. When this formalism is extended to include piezoelectric effects, the matrix function $\mathbf{M}(\mu)$ that governs the (reduced) eigenvector has the dimension 3×3, instead of 2×2. Compared to the 2-D anisotropic elasticity problem, the number of distinct types of eigenvalues increases from 5 to 11, and the number of distinct classes of materials increases from 5 to 14. Each distinct class of material has a peculiar representation of the general solution in terms of four complex conjugate pairs of eigensolutions of the zeroth or higher orders. None of these 14 types can be eliminated, i.e., replaced entirely by the others. Some eigensolutions of degenerate piezoelectric materials require new analytical forms of expression that are not found in 2-D elasticity (Yin, 2000a) or in unsymmetric laminated plate theory (Yin, 2003a, 2000b). Thus, although the general scheme of the present investigation parallels similar developments in these two simpler theories, the details of the case-by-case analysis are considerably different, leading to a greater variety of the types of eigenvalues and classes of materials, and algebraic complexity of the solution spaces. The main results of the present analysis are summarized in Theorems 1–9 and Appendix B of the paper, which may provide a quick reference to the readers who are familiar with the analogous developments in 2-D elasticity and coupled anisotropic plate theory.

2. Governing equations: eigenvalues and zeroth-order eigensolutions

For an anisotropic piezoelectric material, let the components of the strain and stress tensors be assembled in 1-D arrays

$$\{\varepsilon\} = \{\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\}^{\mathrm{T}}, \quad \{\sigma\} = \{\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}\}^{\mathrm{T}}. \tag{2.1}$$

Furthermore, let the electric field and the electric displacement be denoted by

$$\{E\} = \{-\phi_{,x}, -\phi_{,y}, -\phi_{,z}\}^{\mathrm{T}}, \quad \{D\} = \{D_x, D_y, D_z\}^{\mathrm{T}}, \tag{2.2}$$

where ϕ is the electric potential, and the superscripts 'T' indicate taking the transpose of a matrix or a vector. The constitutive relations of a linearly piezoelectric material are often given as follows:

$$\{\sigma\} = [C]_{6\times6}\{\varepsilon\} - [L]^{\mathrm{T}}\{E\}, \quad \{D\} = [L]_{3\times6}\{\varepsilon\} + [K]_{3\times3}\{E\}, \tag{2.3}$$

where the matrices $[C]$, $[L]$ and $[K]$ consist of the elastic constants, the piezoelectric constants and dielectric constants, respectively. The subscripts of the matrices show their dimensions.

In this work, we restrict attention to 2-D solutions, for which all components in Eqs. (2.1) and (2.2) depend only on two coordinates x and y. Then the equation $\operatorname{div}\{D\} = 0$ becomes $D_{x,x} + D_{y,y} = 0$ and, since D_x and D_y are independent of z, there is a scalar function $\varsigma(x, y)$ such that

$$D_x = \varsigma_{,y}, \quad D_y = -\varsigma_{,x}. \tag{2.4}$$

Since all components of $\{E\}$ are independent of z, one has $-\phi_{,xz} = -\phi_{,yz} = -\phi_{,zz} = 0$, so that

$$-\phi_{,z} = E_z = \text{constant}. \tag{2.5a}$$

When all components of the strain are independent of z, the differential equations of compatibility imply that ε_z must be a linear function of both x and y

$$\varepsilon_z(x, y) = \varepsilon_0 + x\varepsilon_1 + y\varepsilon_2. \tag{2.5b}$$

Analogous to the case of two-dimensional elasticity, the determination of the general solution may be drastically simplified by using the inverse form of the preceding constitutive relation, i.e.,

$$\{\varepsilon\} = [S]_{6\times6}\{\sigma\} + [R]^{\mathrm{T}}\{D\}, \quad -\{E\} = [R]_{3\times6}\{\sigma\} - [H]_{3\times3}\{D\}, \tag{2.6a,b}$$

$$\begin{bmatrix} [S]_{6\times6} & [R]^{\mathrm{T}} \\ [R]_{3\times6} & -[H]_{3\times3} \end{bmatrix} = \begin{bmatrix} [C] & [L]^{\mathrm{T}} \\ [L] & -[K] \end{bmatrix}^{-1}. \tag{2.7}$$

If there is no piezoelectric coupling, then $[L]_{3\times6}$ and $[R]_{3\times6}$ are null matrices. Furthermore, $[S] = [C]^{-1}$ and $[H] = [K]^{-1}$. When coupling exists, $[S]$ and $[C]$ are *not* the inverses of each other and, if $[C]$ is regarded as the elastic stiffness matrix, $[S]$ is not properly a compliance matrix since it also involves the elements of $[K]$ and $[L]$.

In view of Eq. (2.5), the third rows of Eq. (2.6a,b) become, respectively

$$\sum S_{3j}\sigma_j + \sum R_{k3}D_k = \varepsilon_z, \quad -\sum R_{3j}\sigma_j + \sum H_{3k}D_k = E_z. \tag{2.8}$$

These two equations may be solved to express σ_z and D_z in terms of ε_z, E_z and other components of $\{\sigma\}$ and $\{D\}$. By substituting the resulting expressions into all equations of (2.6a) except the third one, and into the first and second rows of Eq. (2.6b), then, after rearranging the order of equations and variables, one obtains the constitutive relations for two-dimensional fields in the following form:

$$\{\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}\}^{\mathrm{T}} = [\beta]\{\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}\}^{\mathrm{T}} + [\gamma]^{\mathrm{T}}\{\varsigma_{,y}, -\varsigma_{,x}\}^{\mathrm{T}} + [\lambda]\{\varepsilon_z, E_z\}^{\mathrm{T}}, \tag{2.9a}$$

$$\{\phi_{,x}, \phi_{,y}\}^{\mathrm{T}} = [\gamma]\{\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}\}^{\mathrm{T}} - [\alpha]\{\varsigma_{,y}, -\varsigma_{,x}\}^{\mathrm{T}} + [\kappa]\{\varepsilon_z, E_z\}^{\mathrm{T}}, \tag{2.9b}$$

where the constant matrices $[\beta]$, $[\gamma]$ and $[\alpha]$ are expressed by

$$[\alpha] = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{11} & \alpha_{22} \end{bmatrix}, \quad [\gamma] = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{16} & \gamma_{15} & \gamma_{14} \\ \gamma_{12} & \gamma_{22} & \gamma_{26} & \gamma_{25} & \gamma_{44} \end{bmatrix},$$

$$[\beta] = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{16} & \beta_{15} & \beta_{14} \\ \beta_{12} & \beta_{22} & \beta_{26} & \beta_{25} & \beta_{24} \\ \beta_{16} & \beta_{26} & \beta_{66} & \beta_{56} & \beta_{46} \\ \beta_{15} & \beta_{25} & \beta_{56} & \beta_{55} & \beta_{45} \\ \beta_{14} & \beta_{24} & \beta_{46} & \beta_{45} & \beta_{44} \end{bmatrix}. \tag{2.10a,b,c}$$

Notice that the rows and columns of $[\beta]$, and the columns of $[\gamma]$, are not marked by consecutive indices from 1 to 5, but rather marked by the sequence of indices 1, 2, 6, 5 and 4 that correspond to the positions of the elements ε_x, ε_y, γ_{xy}, γ_{xz} and γ_{yz} in the six-dimensional array $\{\varepsilon\}$ of Eq. (2.1) before the rearrangement. This ordering of indices is adopted so that, when the piezoelectric effect is absent, the present formulation reduces without change to the notation of Lekhnitskii's formalism for two-dimensional anisotropic elasticity (Lekhnitskii, 1963).

For a given constant E_z and a given bilinear function $\varepsilon_z(x,y) = \varepsilon_0 + x\varepsilon_1 + y\varepsilon_2$, any solution of Eq. (2.9a,b) may be separated into a particular solution and a complementary solution. The latter is a solution of Eq. (2.9a,b) with $E_z = 0$ and $\varepsilon_z(x,y) \equiv 0$. Particular solutions are not unique, and we may choose one such that $\{\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}\}^{\mathrm{T}} = \mathbf{0}$. Then Eq. (2.9b) reduces to

$$\{\phi_{,x}, \phi_{,y}\}^{\mathrm{T}} + [\alpha]\{\varsigma_{,y}, -\varsigma_{,x}\}^{\mathrm{T}} = [\kappa]\{\varepsilon_0 + x\varepsilon_1 + y\varepsilon_2, E_z\}^{\mathrm{T}},$$

which is clearly satisfied by an appropriate choice of the two quadratic functions $\phi(x,y)$ and $\varsigma(x,y)$. Eq. (2.9a) yields the strains of the particular solution

$$\{\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}\}^{\mathrm{T}} = [\gamma]^{\mathrm{T}}\{\varsigma_{,y}, -\varsigma_{,x}\}^{\mathrm{T}} + [\lambda]\{\varepsilon_z, E_z\}^{\mathrm{T}}.$$

Since the strains are linear in the coordinates, they automatically satisfy the compatibility equations. This particular solution also trivially satisfies the equilibrium equation because all components of stress vanish except for σ_z, and the latter is independent of z. This implies the following theorem:

Theorem 1. *The general solution of two-dimensional linear piezoelectricity may be separated into* (*i*) *a particular solution that has vanishing* $\{\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}\}$, *a constant z-directional electric field* E_z, *and a linearly varying* ε_z, *as shown by Eq.* (2.5a,b), *and* (*ii*) *a complementary solution which has vanishing* E_z *and* ε_z *in the entire domain. The particular solution satisfies equilibrium of stress*, *compatibility of strain*, *and the constitutive equations* (2.9a) *and* (2.9b), *and so must the complementary solution.*

In the following, we always assume that a particular solution has been found in this or other manner, and the attention will be restricted to the complementary solution, for which $\{\varepsilon_z, E_z\}^{\mathrm{T}}$ vanishes in Eqs. (2.9a,b).

In the absence of body forces, the equilibrium equations of the stress field imply that the five stress components of Eq. (2.9a,b) may be represented by the derivatives of a pair of stress functions $F(x,y)$ and $\Psi(x,y)$

$$\sigma_x = F_{,yy}, \quad \sigma_y = F_{,xx}, \quad \tau_{xy} = -F_{,xy}, \quad \tau_{xz} = \Psi_{,y}, \quad \tau_{yz} = -\Psi_{,x}. \tag{2.11}$$

We seek solutions for the displacements $\{u, v, w\}$, the stress potentials $\{F_y, -F_x, \Psi\}$, the potential $\phi(x,y)$ of the electric field and the skew potential $\varsigma(x,y)$ of the electric displacement, in the following form:

$$\boldsymbol{\chi} = \{F_{,y}, -F_{,x}, \Psi, \varsigma, u, v, w, \phi\}^{\mathrm{T}} = f(z, \mu_0)\boldsymbol{\xi}, \quad z \equiv x + \mu_0 y, \tag{2.12}$$

where $\boldsymbol{\xi}$ is a complex constant vector. The scalar function f is analytic in the first argument z, and the complex parameter μ_0 affects f explicitly as the second argument and implicitly through z.

The complex parameter μ_0 and the complex vector $\boldsymbol{\xi}$ will be identified as the material eigenvalues and eigenvectors. The strain, stress and electric fields associated with (2.12) are easily obtained by differentiation. Since $\tau_{xy} = -(F_{,y})_{,x} = -\xi_1 f_{,z}(z, \mu_0) = (-F_{,x})_{,y} = \xi_2 \mu_0 f_{,z}(z, \mu_0)$, one has

$$\xi_1 = -\mu_0 \xi_2. \tag{2.13}$$

Eqs. (2.11)–(2.13), in conjunction with the strain–displacement relation, yield

$$\boldsymbol{\theta} \equiv \{\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}, \phi_{,x}, \phi_{,y}\}^{\mathrm{T}} = f_{,z}(z, \mu)\boldsymbol{\Theta}(\mu)\{\xi_5, \xi_6, \xi_7, \xi_8\}^{\mathrm{T}}, \tag{2.14a}$$

$$\boldsymbol{\psi} \equiv \{F_{,yy}, F_{,xx}, -F_{,xy}, \Psi_{,y}, -\Psi_{,x}, \varsigma_{,y}, -\varsigma_{,x}\}^{\mathrm{T}} = f_{,z}(z, \mu)\boldsymbol{\Psi}(\mu)\{\xi_2, \xi_3, \xi_4\}^{\mathrm{T}}, \tag{2.14b}$$

where the right-hand sides are evaluated at $\mu = \mu_0$ and where the matrix functions $\boldsymbol{\Theta}(\mu)$ and $\boldsymbol{\Psi}(\mu)$ are defined by

$$\boldsymbol{\Theta}(\mu) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ \mu & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \mu \end{bmatrix}, \quad \boldsymbol{\Psi}(\mu) = \begin{bmatrix} -\mu^2 & 0 & 0 \\ -1 & 0 & 0 \\ \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mu \\ 0 & 0 & -1 \end{bmatrix}. \tag{2.15a,b}$$

Notice that each column of $\boldsymbol{\Psi}(\mu)$ is orthogonal to all columns of $\boldsymbol{\Theta}(\mu)$. Consequently

$$\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\Theta} = \mathbf{0}, \quad \boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{\Psi} = \mathbf{0}. \tag{2.16a,b}$$

Eqs. (2.14), (2.15) and (2.9a,b) with vanishing E_z and ε_z yield

$$\boldsymbol{\Theta}(\mu_0)\{\xi_5, \xi_6, \xi_7, \xi_8\}^{\mathrm{T}} = \begin{bmatrix} [\beta] & [\gamma]^{\mathrm{T}} \\ [\gamma] & -[\alpha] \end{bmatrix} \boldsymbol{\Psi}(\mu_0)\{\xi_2, \xi_3, \xi_4\}^{\mathrm{T}}. \tag{2.17}$$

Pre-multiplication of the last equation by $\boldsymbol{\Psi}(\mu_0)^{\mathrm{T}}$ gives

$$\mathbf{M}(\mu_0)\boldsymbol{\eta} = \mathbf{0}, \tag{2.18}$$

where

$$\boldsymbol{\eta} \equiv \{\xi_2, \xi_3, \xi_4\}^{\mathrm{T}}, \quad \mathbf{M}(\mu) \equiv \boldsymbol{\Psi}(\mu)^{\mathrm{T}}[\varpi]\boldsymbol{\Psi}(\mu), \quad [\varpi] \equiv \begin{bmatrix} [\beta] & [\gamma]^{\mathrm{T}} \\ [\gamma] & -[\alpha] \end{bmatrix}. \tag{2.19a,b,c}$$

The 3×3 symmetric matrix function $\mathbf{M}(\mu)$ has the components

$$\begin{aligned} M_{11} &= \beta_{22} - 2\beta_{26}\mu + (2\beta_{12} + \beta_{66})\mu^2 - 2\beta_{16}\mu^3 + \beta_{11}\mu^4, \\ M_{12} &= \beta_{24} - (\beta_{25} + \beta_{46})\mu + (\beta_{14} + \beta_{56})\mu^2 - \beta_{15}\mu^3, \\ M_{22} &= \beta_{44} - 2\beta_{45}\mu + \beta_{55}\mu^2, \quad M_{13} = \gamma_{22} - (\gamma_{12} + \gamma_{26})\mu + (\gamma_{12} + \gamma_{16})\mu^2 - \gamma_{11}\mu^3, \\ M_{23} &= \gamma_{24} - (\gamma_{14} + \gamma_{25})\mu + \gamma_{15}\mu^2, \quad M_{33} = -(\alpha_{22} - 2\alpha_{12}\mu + \alpha_{11}\mu^2). \end{aligned} \tag{2.20}$$

Eq. (2.18) has a nontrivial solution for $\boldsymbol{\eta}$ if and only if μ_0 is a root of the characteristic equation

$$\delta(\mu) \equiv |\mathbf{M}(\mu)| = 0. \tag{2.21}$$

We define

$$\mathbf{J}_1(\mu) = \begin{bmatrix} -\mu & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{J}_3(\mu) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mu & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.22a,b}$$

$$\mathbf{J}_2(\mu) = \mathbf{J}_3(\mu)[\varpi]\mathbf{\Psi}(\mu), \quad \mathbf{J}(\mu) = \begin{bmatrix} \mathbf{J}_1(\mu) \\ \mathbf{J}_2(\mu) \end{bmatrix}. \tag{2.22c,d}$$

Notice that $\mathbf{J}_3\mathbf{\Theta} = \mathbf{I}_4$. Premultiplying Eq. (2.22c) by $\mathbf{\Theta}(\mu)$, one obtains, after some algebraic manipulation

$$\mathbf{\Theta}(\mu)\mathbf{J}_2(\mu) = [\varpi]\mathbf{\Psi}(\mu) + \begin{bmatrix} 0 & 0 & 0 \\ M_{11}(\mu) & M_{12}(\mu) & M_{13}(\mu) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ M_{12}(\mu) & M_{22}(\mu) & M_{23}(\mu) \\ 0 & 0 & 0 \\ M_{13}(\mu) & M_{23}(\mu) & M_{33}(\mu) \end{bmatrix} \tag{2.23}$$

Hence, taking $\boldsymbol{\eta}$ to be a nontrivial solution of Eq. (2.18), the eight components of

$$\boldsymbol{\xi} = \mathbf{J}(\mu_0)\boldsymbol{\eta} \tag{2.24}$$

satisfy Eqs. (2.13) and (2.17), and therefore yield a piezoelectric equilibrium solution of the strain, stress and the electric fields via Eq. (2.14a,b). The roots of the characteristic equation (2.21) are the *eigenvalues* of the material, and the associated vectors $\boldsymbol{\xi}$ are called *zeroth-order eigenvectors*. The eigenvalues occur in complex conjugate pairs because the characteristic equation has real coefficients. Furthermore, $\bar{\boldsymbol{\xi}}$, the complex conjugate of $\boldsymbol{\xi}$, is an eigenvector associated with the conjugate eigenvalue $\bar{\mu}_0$. Then, according to Eq. (2.12), for any complex analytic function $f(z, \mu_0)$,

$$\boldsymbol{\chi} + \bar{\boldsymbol{\chi}} = f(z, \mu_0)\boldsymbol{\xi} + f(\bar{z}, \bar{\mu}_0)\bar{\boldsymbol{\xi}} = 2Re[f(z, \mu_0)\boldsymbol{\xi}] \tag{2.25}$$

yields real-valued $F_{,y}$, $-F_x$, Ψ, ς, u, v, w and ϕ, whose derivatives, as shown by Eq. (2.14a,b), satisfy the piezoelectric constitutive relations, the stress equilibrium equations, and the compatibility of strain. The complex-valued function $\boldsymbol{\chi} = f(z, \mu_0)\boldsymbol{\xi}$ will be called a *zeroth-order eigensolution* associated with the eigenvalue μ_0 (in contrast to the higher-order eigensolutions, to be introduced in the next section, which involve the μ-derivatives of the arbitrary function $f(z, \mu)$ of the various orders in addition to the function itself). We now give a proof of the complexity of the eigenvalues different from that given previously by Suo et al. (1992).

Theorem 2. *The eigenvalues cannot be real if the material has a positive-definite energy density.*

Proof. Suppose that Eq. (2.21) has a real root μ_0 then $M(\mu_0)$ is a *real*, singular matrix and Eq. (2.18) must possess a real, nontrivial solution $\boldsymbol{\eta} = \{\xi_2, \xi_3, \xi_4\}^{\mathrm{T}}$ (if $\boldsymbol{\eta}$ is complex then both the real and imaginary parts of $\boldsymbol{\eta}$ are also solutions) which yields a real eigenvector $\boldsymbol{\xi} = \mathbf{J}(\mu_0)\boldsymbol{\eta}$ because $\mathbf{J}(\mu_0)$ is also a real matrix. The choice $f \equiv x + \mu_0 y$ gives $f_{,z} \equiv 1$ and the real eigensolution $\boldsymbol{\chi} = (x + \mu_0 y)\boldsymbol{\xi}$. For this eigensolution, Eqs. (2.14a,b) and (2.17) imply that the energy density function vanishes

$$1/2(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz} + D_xE_x + D_yE_y) = 1/2\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\theta} = 1/2\boldsymbol{\eta}^{\mathrm{T}}\mathbf{M}(\mu_0)\boldsymbol{\eta} = 0$$

for a nontrivial field $\boldsymbol{\psi}$ of the stresses and the electric displacement:

$$\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\psi} = \boldsymbol{\eta}^{\mathrm{T}}\boldsymbol{\Psi}(\mu_0)^{\mathrm{T}}\boldsymbol{\Psi}(\mu_0)\boldsymbol{\eta} = (\mu_0^4 + \mu_0^2 + 1)\xi_2^2 + (\mu_0^2 + 1)(\xi_3^2 + \xi_4^2) > 0.$$

In the following, we assume that the energy density function is positive definite. Then there can be no real eigenvalues. All eigenvalues may be grouped into two sets, where the first set $\{\mu\}_\perp$ consists of four eigenvalues (not necessarily all distinct) with positive imaginary parts, and the second set, $\{\bar{\mu}\}_\perp$, is the complex conjugate of $\{\mu\}_\perp$.

Another important property is the orthogonality (to be defined shortly) of the eigenvectors associated with any pair of distinct eigenvalues. If $\mu \neq \mu'$, then Eq. (2.22a,b) yield

$$\mathbf{J}_1(\mu')^{\mathrm{T}}\mathbf{J}_2(\mu) = (\mu' - \mu)^{-1}\{\boldsymbol{\Psi}(\mu')^{\mathrm{T}} - \boldsymbol{\Psi}(\mu)^{\mathrm{T}}\}[\varpi]\boldsymbol{\Psi}(\mu).$$

Hence, due to the symmetry of $[\varpi]$, one has

$$\begin{aligned}
&\mathbf{J}_1(\mu')^{\mathrm{T}}\mathbf{J}_2(\mu) + \mathbf{J}_2(\mu')^{\mathrm{T}}\mathbf{J}_1(\mu)\\
&\quad = (\mu' - \mu)^{-1}\{\boldsymbol{\Psi}(\mu')^{\mathrm{T}}[\varpi]\boldsymbol{\Psi}(\mu') - \boldsymbol{\Psi}(\mu)^{\mathrm{T}}[\varpi]\boldsymbol{\Psi}(\mu') + \boldsymbol{\Psi}(\mu')^{\mathrm{T}}[\varpi]\boldsymbol{\Psi}(\mu) - \boldsymbol{\Psi}(\mu)^{\mathrm{T}}[\varpi]\boldsymbol{\Psi}(\mu)\}\\
&\quad = (\mu' - \mu)^{-1}\{\mathbf{M}(\mu') - \mathbf{M}(\mu)\}.
\end{aligned} \tag{2.26}$$

If μ and μ' are distinct eigenvalues, with $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$ as the corresponding solutions of Eq. (2.18), then Eq. (2.26) implies that the eigenvectors $\boldsymbol{\xi} = \mathbf{J}(\mu)\boldsymbol{\eta}$ and $\boldsymbol{\xi}' = \mathbf{J}(\mu')\boldsymbol{\eta}'$ satisfy

$$[\![\boldsymbol{\xi}', \boldsymbol{\xi}]\!] \equiv \boldsymbol{\xi}'^{T}\mathbf{II}\boldsymbol{\xi} = \boldsymbol{\eta}'^{T}\{\mathbf{J}_1(\mu')^{\mathrm{T}}\mathbf{J}_2(\mu) + \mathbf{J}_2(\mu')^{\mathrm{T}}\mathbf{J}_1(\mu)\}\boldsymbol{\eta} = 0, \tag{2.27}$$

where

$$\mathbf{II} \equiv \begin{bmatrix} \mathbf{0}_{4\times4} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{0}_{4\times4} \end{bmatrix} \tag{2.28}$$

and $\mathbf{I}_k$ and $\mathbf{0}_{k\times k}$ denote, respectively, identity and zero matrices of dimension $k\times k$. Hence any two zeroth-order eigenvectors $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ associated with distinct eigenvalues are orthogonal in the sense of Eq. (2.27). In particular, $\boldsymbol{\xi}$ is orthogonal to its complex conjugate vector:

$$[\![\boldsymbol{\xi}, \bar{\boldsymbol{\xi}}]\!] = 0. \tag{2.29}$$

In the following, $[\![\boldsymbol{\xi}', \boldsymbol{\xi}]\!] = [\![\boldsymbol{\xi}, \boldsymbol{\xi}']\!]$ will be called the *binary product* of the two vectors $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$. It will be shown later that Eq. (2.29) is also valid if one or both vectors $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ are higher-order eigenvectors associated with different *multiple* eigenvalues. Therefore, the eigenvectors belonging to an eigenvalue of multiplicity $p(1 \leqslant p \leqslant 4)$ span a p-dimensional subspace which is orthogonal to the subspaces of other simple or multiple eigenvalues. Hence the eight dimensional complex vector space is the direct sum of a number of orthogonal subspaces, one for each distinct eigenvalue. In this work, orthogonality of $\boldsymbol{\xi}$ is always defined in the sense of the binary product, whereas orthogonality of 3-D complex vectors (between $\boldsymbol{\eta}$ and the column vectors of $\mathbf{M}(\mu)$, for example) is with regard to the usual scalar product.

3. Normal, abnormal and superabnormal eigenvalues: zeroth-order eigensolutions

In two-dimensional piezoelectricity, the mathematical structure of the solution space, and the analytical form of the general solution for the different classes of materials, depend essentially on the multiplicity of the eigenvalues, and on the three matrices $\mathbf{M}$, $\mathbf{W}$ and $\mathbf{W}'$ evaluated at these eigenvalues. Here $\mathbf{W}$ denotes the adjoint matrix of $\mathbf{M}$, and $\mathbf{W}'$ the derivative of $\mathbf{W}$. These distinctions have profound effects, as seen in the related previous works on anisotropic elasticity and laminated plates (Yin, 2000a,b and Yin, 2003a), and in the results concerning the structure and singularity of Green's functions (Yin, 2004, 2005, in press a, in press b). Since the various classes of piezoelectric materials have different explicit expressions of the general solution, it follows that Green's functions of finite and infinite domains, integral representations of the interior solutions in terms of the boundary data, and singularity solutions of cracks and multi-material wedges all assume fundamentally different mathematical forms depending on the classes of materials.

All eigenvalues μ_0 of the present problem will be classified into three major types depending on the rank of the matrix $\mathbf{M}(\mu_0)$:

(1) Normal eigenvalue, for which $\mathbf{M}(\mu_0)$ is of rank 2.
(2) Abnormal eigenvalue, for which $\mathbf{M}(\mu_0)$ is of rank 1. Then the adjoint matrix $\mathbf{W}(\mu_0)$ must be null. If the multiplicity exceeds 2, this types is further divided into two subclasses:
Abnormal-α: $\mathbf{W}'(\mu_0)$ is not null;
Abnormal-β: $\mathbf{W}'(\mu_0)$ is a null matrix.
(3) Superabnormal eigenvalue, for which $\mathbf{M}(\mu_0)$ is a null matrix.

Notice that Eq. (2.18) of the present problem involves a 3×3 matrix $\mathbf{M}(\mu)$, whereas the corresponding $\mathbf{M}$ matrix in the related problems of 2-D anisotropic elasticity, and of anisotropic plates with bending-stretching coupling, has the dimension 2×2 (Yin, 2000a, 2003a,b). A $2\times$ 2 matrix has the peculiar property that it is null if and only if its adjoint matrix is. Not so for a 3×3 matrix $\mathbf{M}(\mu)$ in piezoelectricity, which also yields some new forms of eigensolutions that are not found in 2-D elasticity and coupled anisotropic plate theory.

The matrix function $\mathbf{M}(\mu)$ and its adjoint $\mathbf{W}(\mu)$ satisfy the polynomial identity

$$\mathbf{M}(\mu)\mathbf{W}(\mu) = \mathbf{W}(\mu)\mathbf{M}(\mu) = \delta(\mu)\mathbf{I}_3. \tag{3.1}$$

Differentiation yields

$$\sum_{0\leqslant k\leqslant n}(n,k)\mathbf{M}^{(n-k)}(\mu)\mathbf{W}^{(k)}(\mu) = \delta^{(n)}(\mu)\mathbf{I}_3 \quad (n=1,2,\ldots). \tag{3.2}$$

If μ_0 is not normal, then $\mathbf{W}(\mu_0)=\mathbf{0}$, so that Eq. (3.2) with $n=1$ reduces to $\mathbf{M}(\mu_0)\mathbf{W}'(\mu_0)=\delta'(\mu_0)\mathbf{I}_3$ and, since $\mathbf{M}(\mu_0)$ is a singular matrix, one must have $\delta'(\mu_0)=0$. If μ_0 is superabnormal, then the matrix $\mathbf{M}(\mu_0)$ is null, and so must be $\mathbf{W}(\mu_0)$. From Eq. (3.2) one has $\mathbf{M}\mathbf{W}'=\mathbf{0}$ and $2\mathbf{M}'\mathbf{W}'=\delta''\mathbf{I}_3$. Hence $\mathbf{W}'(\mu_0)$ is a singular matrix and $\delta''(\mu_0)=0$. These results imply that the multiplicity of abnormal and superabnormal eigenvalues are, respectively, at least 2 and 3.

Two related types of mathematical expressions are often used together in this paper: those involving scalar and matrix *functions* of the complex parameter μ, and the others involving the *values* of these functions at an eigenvalue μ_0. It is convenient to adopt a special notation $\mathbf{\Phi}|_0$ to denote the value of a simple or compound expression $\mathbf{\Phi}$ at $\mu=\mu_0$.

3.1. Zeroth-order eigenvector of a normal eigenvalue

For a simple or multiple *normal* eigenvalue μ_0, $\mathbf{M}(\mu_0)$ has two independent columns. Then Eq. (2.18) has only one independent solution $\boldsymbol{\eta}$, because $\boldsymbol{\eta}$ must be orthogonal to the two independent columns of $\mathbf{M}(\mu_0)$. Evaluating Eq. (3.1) at $\mu=\mu_0$ one has

$$\mathbf{M}\mathbf{W}|_0 = \mathbf{W}\mathbf{M}|_0 = \mathbf{0}_{3\times 3}.$$

Hence the nonvanishing columns of $\mathbf{W}(\mu_0)$ are all proportional to $\boldsymbol{\eta}$, so that $\mathbf{W}(\mu_0)$ is of rank one. Not all diagonal elements of $\mathbf{W}(\mu_0)$ may vanish because otherwise the off-diagonal elements $W_{ij}=\surd(W_{ii}W_{jj})$

(where $i \neq j$ and no summation is implied for the repeated indices) would also vanish and $\mathbf{W}(\mu_0)$ would be the null matrix, contradicting the assumption that $\mathbf{M}(\mu_0)$ is of rank 2. If the k th diagonal element W_{kk} does not vanish, we let $\boldsymbol{\rho}_k$ and $\boldsymbol{\eta}$ be, respectively, the kth column of $\mathbf{I}_3$ and $\mathbf{W}(\mu_0)$. Then

$$\boldsymbol{\eta} \equiv \mathbf{W}(\mu_0)\boldsymbol{\rho}_k, \quad W_{kk} = \boldsymbol{\rho}_k^{\mathrm{T}}\mathbf{W}(\mu_0)\boldsymbol{\rho}_k, \quad \mathbf{W}(\mu_0) = \boldsymbol{\eta}\boldsymbol{\eta}^{\mathrm{T}}/W_{kk}. \tag{3.3a,b,c}$$

Postmultiplication of Eq. (3.2) by $\mathbf{W}(\mu_0)$ yields

$$\mathbf{W}\mathbf{M}'\mathbf{W}|_0 = \delta'\mathbf{W}|_0 = \{\delta'(\mu_0)/W_{kk}\}\boldsymbol{\eta}\boldsymbol{\eta}^{\mathrm{T}}.$$

This matrix equation has only one nonzero element at the kth diagonal position, i.e.,

$$\boldsymbol{\eta}^{\mathrm{T}}\mathbf{M}'(\mu_0)\boldsymbol{\eta} = W_{kk}\delta'(\mu_0),$$

If μ_0 is a simple eigenvalue, then $\delta(\mu_0) = 0$ but $\delta'(\mu_0) \neq 0$, and the eigenvector $\boldsymbol{\xi} = \mathbf{J}(\mu_0)\boldsymbol{\eta}$ has a nonvanishing binary product with itself:

$$[\![\boldsymbol{\xi}, \boldsymbol{\xi}]\!] = \boldsymbol{\eta}^{\mathrm{T}}\{\mathbf{J}_1(\mu_0)^{\mathrm{T}}\mathbf{J}_2(\mu_0) + \mathbf{J}_2(\mu_0)^{\mathrm{T}}\mathbf{J}_1(\mu_0)\}\boldsymbol{\eta} = \boldsymbol{\eta}^{\mathrm{T}}\mathbf{M}'(\mu_0)\boldsymbol{\eta} = W_{kk}\delta'(\mu_0). \tag{3.4}$$

If the four eigenvalues in $\{\mu\}_\perp$ are all distinct, then eight zeroth-order eigenvectors may be obtained, and they are mutually orthogonal in the sense of Eq. (2.27). Let $\mathbf{Z}$ be the 8×8 *base matrix* containing the eigenvectors as the column vectors, such that the submatrix $\mathbf{Z}_\perp$ of the first four columns are eigenvectors associated with the eigenvalues in $\{\mu\}_\perp$, and the last four columns are the complex conjugates of the first four. Orthogonality of the eigenvectors implies that

$$\boldsymbol{\Omega} \equiv \mathbf{Z}^{\mathrm{T}}\mathbf{I}\mathbf{I}\mathbf{Z} \tag{3.5}$$

is a diagonal matrix, in which all the diagonal elements do not vanish and they have the form of Eq. (3.4). Hence $\mathbf{Z}^{\mathrm{T}}\mathbf{I}\mathbf{I}\mathbf{Z}$ is nonsingular and so must be $\mathbf{Z}$. It follows that the eight eigenvectors associated with distinct eigenvalues are *independent*. Each eigenvector, when multiplied by an arbitrary analytic function, gives an eigensolution in the form of Eq. (2.12).

Notice that in a complex vector space, orthogonality of a set of vectors does not ensure their independence. For example, the scalar product of $\{0, 1, i\}$ and $\{0, i, -1\}$ vanishes, and $\{0, 0, 1, i, 0, 0, 1, i\}$ and $\{0, 0, i, -1, 0, 0, i, -1\}$ are orthogonal in the sense of the binary product, but the two vectors in each pair are also proportional, and thus linearly dependent. Hence the independence of the eight eigenvectors cannot be deduced merely from their orthogonality, Eq. (2.27). One must also use Eq. (3.4) to ensure that each nontrivial eigenvector has a nonvanishing binary product with itself, which implies the invertibility of $\boldsymbol{\Omega}$, and the latter in turn implies the invertibility of $\mathbf{Z}$.

3.2. Zeroth-order eigenvectors of abnormal and superabnormal eigenvalues

If a multiple eigenvalue μ_0 is abnormal or superabnormal, then $\mathbf{M}(\mu_0)$ has at most one independent column. The adjoint matrix $\mathbf{W}(\mu_0)$ is null, so that it has no nontrivial column to be used as the $\boldsymbol{\eta}$-vector for obtaining a zeroth-order eigenvector. The latter must be obtained in different ways. The results are also substantially different, and depend essentially on the type of eigenvalues. With $\mathbf{W}(\mu_0) = \mathbf{0}$, and $\mathbf{M}(\mu_0)$ singular, Eq. (3.2) becomes, for $n = 1$ and 2

$$\mathbf{M}\mathbf{W}'|_0 = \mathbf{0}, \quad (\mathbf{W}''\mathbf{M} + 2\mathbf{W}'\mathbf{M}')|_0 = \delta''(\mu_0)\mathbf{I}_3. \tag{3.6a,b}$$

For an abnormal μ_0, $\mathbf{M}(\mu_0)$ is a symmetric matrix of rank one, so that it has at least one nonzero diagonal element, say, v_k in the kth column. Let $\boldsymbol{\rho}_k$ and $\mathbf{v}$ denote, respectively, the kth column of $\mathbf{I}_3$ and of $\mathbf{M}(\mu_0)$. Then

$$\mathbf{v} = \mathbf{M}(\mu_0)\boldsymbol{\rho}_k, \quad \mathbf{M}(\mu_0) = \mathbf{v}\mathbf{v}^{\mathrm{T}}/v_k \tag{3.7a,b}$$

and Eq. (3.6a) implies that $\mathbf{W}'(\mu_0)$ is singular.

Consider any two *independent* three-dimensional vectors $\boldsymbol{\eta}_1$, and $\boldsymbol{\eta}_2$, both orthogonal to $\mathbf{v}$. Eq. (3.7b) implies

$$\mathbf{M}(\mu_0)\boldsymbol{\eta}_k = \mathbf{0} \quad (k = 1, 2)$$

One obtains two zeroth-order eigenvectors $\boldsymbol{\xi}_1 = \mathbf{J}(\mu_0)\boldsymbol{\eta}_1$ and $\boldsymbol{\xi}_2 = \mathbf{J}(\mu_0)\boldsymbol{\eta}_2$, and Eq. (2.12) gives the corresponding eigensolutions. Since $\boldsymbol{\xi}$ contains the three elements of $\boldsymbol{\eta}$ as its second through the fourth elements, linear independence of $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ implies the same for $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. Thus, while a normal eigenvalue, whether simple or multiple, has only one independent zeroth-order eigensolution, each abnormal eigenvalue has two independent zeroth-order eigensolutions. For definiteness, we make a specific choice of the eigenvectors in the following manner. Consider the matrix function

$$\mathbf{U}_k(\mu) \equiv M_{kk}(\mu)\mathbf{I}_3 + \{\mathbf{M}(\mu)\boldsymbol{\rho}_k\}\boldsymbol{\rho}_k^{\mathrm{T}} - \boldsymbol{\rho}_k\boldsymbol{\rho}_k^{\mathrm{T}}\mathbf{M}(\mu) \quad (\text{no sum on } k) \tag{3.8}$$

Then

$$M_{kk}(\mu_0) = v_k \neq 0, \quad \mathbf{U}_k(\mu_0)\boldsymbol{\rho}_k = \mathbf{v}, \quad \mathbf{v}^{\mathrm{T}}\mathbf{U}_k(\mu_0) = (\mathbf{v}^{\mathrm{T}}\mathbf{v})\boldsymbol{\rho}_k^{\mathrm{T}}. \tag{3.9a,b,c}$$

Hence the kth column of $\mathbf{U}_k(\mu_0)$ is $\mathbf{v}$. The other two columns are orthogonal to $\mathbf{v}$, since the vector on the right-hand side of Eq. (3.9c) has zero elements except for the kth. Let the two corresponding columns of the matrix function $\mathbf{U}_k(\mu)$ be denoted by $\boldsymbol{\eta}_1(\mu)$ and $\boldsymbol{\eta}_2(\mu)$. Then, for $k = 1,2,3$, respectively, one has the following expressions for the functions $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$, $(\mathbf{M}\boldsymbol{\eta}_1)'$ and $(\mathbf{M}\boldsymbol{\eta}_2)'$

$k =$	$\boldsymbol{\eta}_1 =$	$\boldsymbol{\eta}_2 =$	$(\mathbf{M}\boldsymbol{\eta}_1)' =$	$(\mathbf{M}\boldsymbol{\eta}_2)' =$	
1	$\{-M_{12}, M_{11}, 0\}^{\mathrm{T}}$	$\{-M_{13}, 0, M_{11}\}^{\mathrm{T}}$	$\{0, W'_{33}, -W'_{23}\}^{\mathrm{T}}$	$\{0, -W'_{23}, W'_{22}\}^{\mathrm{T}}$	(3.10a)
2	$\{M_{22}, -M_{12}, 0\}^{\mathrm{T}}$	$\{0, -M_{23}, M_{22}\}^{\mathrm{T}}$	$\{W'_{33}, 0, -W'_{13}\}^{\mathrm{T}}$	$\{-W'_{13}, 0, W'_{11}\}^{\mathrm{T}}$	(3.10b)
3	$\{M_{33}, 0, -M_{13}\}^{\mathrm{T}}$	$\{0, M_{33}, -M_{23}\}^{\mathrm{T}}$	$\{W'_{22}, -W'_{12}, 0\}^{\mathrm{T}}$	$\{-W'_{12}, W'_{11}, 0\}^{\mathrm{T}}$	(3.10c)

For an abnormal eigenvalue μ_0, $\mathbf{M}(\mu_0)$ has at least one nonzero diagonal element $M_{kk}(\mu_0)$. Then the vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ corresponding to k in Eqs. (3.10a), (3.10b) or (3.10c) yield two independent zeroth-order eigenvectors $\mathbf{J}\boldsymbol{\eta}_1|_0$ and $\mathbf{J}\boldsymbol{\eta}_2|_0$, and no more than two since the subspace orthogonal to $\mathbf{v}$ is two-dimensional.

For a superabnormal eigenvalue μ_0, $\mathbf{M}(\mu_0)$ is the null matrix, so that Eq. (2.18) is trivially satisfied by any $\boldsymbol{\eta}$. If the three columns of $\mathbf{I}_3$ are chosen to be the $\boldsymbol{\eta}$-vectors, then Eq. (2.24) yields three independent zeroth-order eigenvectors given by the three columns of $\mathbf{J}(\mu_0)$. Each eigenvector, when multiplied by an arbitrary analytic function of $x + \mu_0 y$, is an eigensolution of the zeroth order.

Summarizing the results of the preceding analysis, one has the following theorem:

Theorem 3. *The number of independent zeroth-order eigensolutions possessed by a normal, abnormal and superabnormal eigenvalue is exactly one, two and three, respectively. Hence the multiplicities of the eigenvalues may range from these numbers to four.*

4. Higher-order eigensolutions and the derivative rule

4.1. Identities involving $\mathbf{J}(\boldsymbol{\mu})$, $\mathbf{M}(\boldsymbol{\mu})$, $\mathbf{W}(\boldsymbol{\mu})$ and their derivatives

The binary product as defined by Eq. (2.27) for two vectors may be extended to two matrices of row dimension eight, regardless of their column dimensions. Eq. (2.26) assumes the form

$$(\mu - \tilde{u})[\![\mathbf{J}(\mu), \mathbf{J}(\tilde{u})]\!] = (\mu - \tilde{u})\mathbf{J}(\mu)^{\mathrm{T}}\mathbf{I}\!\mathbf{I}\mathbf{J}(\tilde{u}) = \mathbf{M}(\mu) - \mathbf{M}(\tilde{u}). \tag{4.1}$$

Differentiation with respect to μ gives

$$(\mu-\tilde{u})[\![\mathbf{J}'(\mu),\mathbf{J}(\tilde{u})]\!]+[\![\mathbf{J}(\mu),\mathbf{J}(\tilde{u})]\!]=\mathbf{M}'(\mu). \tag{4.2a}$$

Repeated differentiation yields, for $1 \leqslant s \leqslant N-1$

$$\begin{aligned}&\partial_\mu^{N-s}\partial_{\tilde{u}}^{s}\{(\mu-\tilde{u})[\![\mathbf{J}(\mu),\mathbf{J}(\tilde{u})]\!]\}\\&\quad=(\mu-\tilde{u})[\![\mathbf{J}^{(N-s)}(\mu),\mathbf{J}^{(s)}(\tilde{u})]\!]+(N-s)[\![\mathbf{J}^{(N-s-1)}(\mu),\mathbf{J}^{(s)}(\tilde{u})]\!]-s[\![\mathbf{J}^{(N-s)}(\mu),\mathbf{J}^{(s-1)}(\tilde{u})]\!]=\mathbf{0}\end{aligned} \tag{4.2b}$$

and, if $\mu \neq \tilde{u}$

$$\begin{aligned}[\![\mathbf{J}^{(p)}(\mu),\mathbf{J}^{(q)}(\tilde{u})]\!]&=(-1)^{q+1}\sum_{0\leqslant s\leqslant p}(p,s)(s+q)!(-\mu+\tilde{u})^{-(s+q+1)}\mathbf{M}^{(p-s)}(\mu)\\&\quad+(-1)^{p+1}\sum_{0\leqslant t\leqslant q}(q,t)(t+p)!(\mu-\tilde{u})^{-(t+p+1)}\mathbf{M}^{(q-t)}(\tilde{u}).\end{aligned} \tag{4.3}$$

For two equal eigenvalues $\mu=\tilde{u}$, Eq. (4.2a) reduces to

$$[\![\mathbf{J}(\mu),\mathbf{J}(\mu)]\!]=\mathbf{M}'(\mu). \tag{4.4a}$$

Repeated differentiation yields

$$[\![\mathbf{J}^{(N-s)}(\mu),\mathbf{J}^{(s)}(\mu)]\!]=\{s!(N-s)!/(N+1)!\}\mathbf{M}^{(N+1)}(\mu)\quad(0\leqslant s\leqslant N\leqslant 3). \tag{4.4b}$$

Eqs. (3.2) and (4.4b) yield, after lengthy manipulation

$$[\![(\mathbf{JW})^{(r)},(\mathbf{JW})^{(N-r)}]\!]=r!(N-r)!\sum_{0\leqslant k\leqslant r}\{k!(N+k-1)!\}^{-1}\{\delta^{(N+k-1)}\mathbf{W}^{(k)}-\delta^{(k)}\mathbf{W}^{(N+k-1)}\}. \tag{4.5}$$

The relations Eqs. (4.1) and (4.5) will be used repeatedly in the following analysis.

4.2. The derivative rule

From Eq. (2.14a,b) one obtains, by using Eqs. (2.23) and (2.24)

$$\boldsymbol{\theta}-[\varpi]\boldsymbol{\psi}=f_{,z}\mathbf{U}\mathbf{M}(\mu)\boldsymbol{\eta}, \tag{4.6}$$

where $\mathbf{U}$ has the transpose matrix

$$\mathbf{U}^{\mathrm{T}}\equiv\begin{bmatrix}0&1&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&1\end{bmatrix} \tag{4.7}$$

In the preceding derivation of zeroth-order eigensolutions, $\boldsymbol{\eta}$ is taken to be a nontrivial solution of Eq. (2.18) corresponding to an eigenvalue μ_0. In the following investigation of higher-order eigensolutions, $\boldsymbol{\eta}$ is considered tentatively to be a function of μ, and the latter is regarded as a variable. The functional form of $\boldsymbol{\eta}(\mu)$ will be specified later according to the type of the material. Instead of the constant vector of Eq. (2.24), $\boldsymbol{\xi}$ will be redefined as the following *function* of μ

$$\boldsymbol{\xi}(\mu)=\mathbf{J}(\mu)\boldsymbol{\eta}(\mu), \tag{4.8}$$

where $\mathbf{J}(\mu)$ is given by Eq. (2.22). Then Eq. (2.13) is replaced by the relation $\xi_1(\mu)=-\mu\xi_2(\mu)$, and Eq. (2.14a,b) define the functions $\boldsymbol{\theta}=\boldsymbol{\theta}(\mu)$ and $\boldsymbol{\psi}=\boldsymbol{\psi}(\mu)$, which are found to satisfy the relation (4.6) for arbitrary μ. Differentiation of Eq. (4.6) yields additional identities

$$\mathrm{d}\boldsymbol{\theta}/\mathrm{d}\mu-[\varpi]\mathrm{d}\boldsymbol{\psi}/\mathrm{d}\mu=\mathbf{U}\{(\mathrm{d}f_{,z}/\mathrm{d}\mu)\mathbf{M}\boldsymbol{\eta}+f_{,z}(\mathbf{M}\boldsymbol{\eta})'\}, \tag{4.9a}$$

$$\mathrm{d}^2\boldsymbol{\theta}/d\mu^2 - [\varpi]\mathrm{d}^2\boldsymbol{\psi}/\mathrm{d}\mu^2 = \mathbf{U}\{(\mathrm{d}^2 f_{,z}/\mathrm{d}\mu^2)\mathbf{M}\boldsymbol{\eta} + 2(\mathrm{d}f_{,z}/\mathrm{d}\mu)(\mathbf{M}\boldsymbol{\eta})' + f_{,z}(\mathbf{M}\boldsymbol{\eta})''\}, \tag{4.9b}$$

$$\mathrm{d}^3\boldsymbol{\theta}/\mathrm{d}\mu^3 - [\varpi]\mathrm{d}^3\boldsymbol{\psi}/\mathrm{d}\mu^3 = \mathbf{U}\{(\mathrm{d}^3 f_{,z}/\mathrm{d}\mu^3)\mathbf{M}\boldsymbol{\eta} + 3(\mathrm{d}^2 f_{,z}/\mathrm{d}\mu^2)(\mathbf{M}\boldsymbol{\eta})' + 3(\mathrm{d}f_{,z}/\mathrm{d}\mu)(\mathbf{M}\boldsymbol{\eta})'' + f_{,z}(\mathbf{M}\boldsymbol{\eta})'''\}. \tag{4.9c}$$

The zeroth-order eigensolutions of the preceding section are based on a nontrivial solution $\boldsymbol{\eta}$ of $\mathbf{M}(\mu_0)\boldsymbol{\eta} = \mathbf{0}$. Such a nontrivial solution is given by Eq. (3.3a) for a normal eigenvalue, by Eq. (3.10) for an abnormal eigenvalue, and by an arbitrary vector if μ_0 is superabnormal. With $\mathbf{M}(\mu_0)\boldsymbol{\eta} = \mathbf{0}$, Eq. (4.6) reduces to $\boldsymbol{\theta}(\mu_0) - [\varpi]\boldsymbol{\psi}(\mu_0) = \mathbf{0}$, i.e., the two groups of physical variables $\boldsymbol{\theta} = \{\varepsilon_x,\ \varepsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}, \phi_x, \phi_y\}^{\mathrm{T}}$ and $\boldsymbol{\psi} = \{F_{,yy}, F_{,xx}, -F_{,xy}, \Psi_{,y}, -\Psi_{,x},\quad \varsigma_{,y}, -\varsigma_{,x}\}^{\mathrm{T}}$, as obtained by taking the spatial derivatives of $\boldsymbol{\chi} = f(x+\mu_0 y, \mu_0)\mathbf{J}(\mu_0)\boldsymbol{\eta}(\mu_0)$, satisfy the constitutive relation $\boldsymbol{\theta} = [\varpi]\boldsymbol{\psi}$, so that $\boldsymbol{\chi}$ is indeed an eigensolution. To obtain an nth-order eigensolution, one needs to find a suitable function $\boldsymbol{\eta}(\mu)$ which satisfies the following set of $n+1$ relations at $\mu = \mu_0$:

$$\mathbf{M}\boldsymbol{\eta}|_0 = \mathbf{0}, \quad (\mathbf{M}\boldsymbol{\eta})'|_0 = \mathbf{0}, \ldots, (\mathbf{M}\boldsymbol{\eta})^{(n)}|_0 = \mathbf{0}. \quad (n \leqslant 3) \tag{4.10}$$

Substitution of these relations into Eqs. (4.6) and (4.9a–c) gives

$$(\boldsymbol{\theta} - [\varpi]\boldsymbol{\psi})|_0 = \mathbf{0}, \quad (\mathrm{d}\boldsymbol{\theta}/\mathrm{d}\mu - [\varpi]d\boldsymbol{\psi}/d\mu)|_0 = \mathbf{0}, \ldots, (\mathrm{d}^n\boldsymbol{\theta}/\mathrm{d}\mu^n - [\varpi]\mathrm{d}^n\boldsymbol{\psi}/\mathrm{d}\mu^n)|_0 = \mathbf{0}. \tag{4.11}$$

This implies that the constitutive relation is not only satisfied by the pair $\boldsymbol{\theta}|_0$ and $\boldsymbol{\psi}|_0$, but also by the successive pairs of μ-derivatives of $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$ (evaluated at $\mu = \mu_0$) up to the order n. Since the kth μ-derivatives of $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$ are the spatial gradients of $\mathrm{d}^k\boldsymbol{\chi}/\mathrm{d}\mu^k|_0$, the kth equation in (4.11) implies that $\mathrm{d}^k\boldsymbol{\chi}/\mathrm{d}\mu^k|_0$ is a piezoelectric solution. Let this solution be denoted by $\boldsymbol{\chi}^{[k]}$. Then

$$\boldsymbol{\chi}^{[k]} = \mathrm{d}^k(f\boldsymbol{\xi})/\mathrm{d}\mu^k|_0 = \mathrm{d}^k(f\mathbf{J}\boldsymbol{\eta})/\mathrm{d}\mu^k|_0 = \sum_{0\leqslant j\leqslant k}(k,j)f^{(k-j)}(\mathbf{J}\boldsymbol{\eta})^{(j)}|_0. \tag{4.12}$$

This expression involves the μ-derivatives of the function f up to the order k if $\boldsymbol{\eta}(\mu_0)$ is a nontrivial vector. In that case $\boldsymbol{\chi}^{[k]}$ is called a kth-order eigensolution. If $\boldsymbol{\eta}(\mu_0)$ is the null vector, and if $\boldsymbol{\eta}^{(j)}(\mu_0)$ is the lowest-order derivative of $\boldsymbol{\eta}(\mu)$ that does not vanish at $\mu = \mu_0$, then the highest-order derivative of f appearing in $\boldsymbol{\chi}^{[k]}$ is $f^{(k-j)}$ and $\boldsymbol{\chi}^{[k]}$ is called an eigensolution of the order $k-j$. Since the second, third and fourth elements of $\mathbf{J}\boldsymbol{\eta}$ are identical to the three elements of $\boldsymbol{\eta}$, the vector $(\mathbf{J}\boldsymbol{\eta})^{(j)}|_0$ is not null if and only if $\boldsymbol{\eta}^{(j)}(\mu_0)$ is not null. Hence we have the following theorem:

Theorem 4 (The Derivative Rule). *If a function $\boldsymbol{\eta}(\mu)$ satisfies the $n+1$ conditions of Eq.* (4.10), *then for every integer k such that $0 \leqslant k \leqslant n$, $(\mathbf{J}\boldsymbol{\eta})^{(k)}|_0$ is an eigenvector whenever it is not null. The corresponding eigensolution is given by Eq. (4.12). This eigensolution is of the order k if $\boldsymbol{\eta}(\mu_0)$ is not null; it is of the order $k-j$ if $\boldsymbol{\eta}(\mu)$ and its first $j-1$ derivatives all vanish at $\mu = \mu_0$.*

Notice that the zeroth-order eigenvectors and eigenfunctions are given by $(\mathbf{J}\boldsymbol{\eta})|_0$ and $(f\mathbf{J}\boldsymbol{\eta})|_0$, respectively. Theorem 4 states that the higher-order eigenvectors and eigensolutions are obtained by evaluating the μ-derivatives of $\mathbf{J}\boldsymbol{\eta}$ and $f\mathbf{J}\boldsymbol{\eta}$ of the various orders at μ_0. Hence the theorem may be referred to as the *derivative rule* for determining the higher-order eigensolutions of a multiple eigenvalue. While the derivative relation between the zeroth-order and the higher-order eigensolutions is known heuristically in anisotropic elasticity, Theorem 4 points out that the rule is not unconditionally valid. It presupposes the validity of Eq. (4.10) regarding the various derivatives of $\boldsymbol{\eta}(\mu)$, whose satisfaction in turn depends on an appropriate choices of the functional form of $\boldsymbol{\eta}(\mu)$.

Three different kinds of functions $\boldsymbol{\eta}(\mu)$ are needed in the following analysis, one taken from the column vectors of $\mathbf{W}(\mu)$, another taken from the columns of $\mathbf{I}_3$, and the last kind from Eq. (3.10). All eigensolutions of a normal eigenvalue are obtained by using the column vectors of $\mathbf{W}(\mu)$ as $\boldsymbol{\eta}(\mu)$. But for an abnormal or superabnormal eigenvalue of multiplicity p, a combination of different kinds of $\boldsymbol{\eta}$ functions is needed to obtain a complete set of p independent eigensolutions. This combination depends crucially on the type and multiplicity of the eigenvalue. When all independent eigensolutions associated with the simple and multiple eigenvalues are combined, one obtains the general solution unique to a specific type of piezoelectric material.

4.3. Higher-order eigensolutions of a normal eigenvalue

If the characteristic equation (2.21) has a normal eigenvalue μ_0 of multiplicity p, then $\delta(\mu)$ and its derivatives up to the order $p-1$ all vanish at μ_0 but the pth derivative does not. Eqs. (3.1) and (3.2) imply

$$\mathbf{MW}|_0 = \mathbf{0}, \quad (\mathbf{MW}' + \mathbf{M}'\mathbf{W})|_0 = \mathbf{0} \quad \text{if } p \geqslant 2, \tag{4.13a,b}$$

$$(\mathbf{MW}'' + 2\mathbf{M}'\mathbf{W}' + \mathbf{M}''\mathbf{W})|_0 = \mathbf{0} \quad \text{if } p \geqslant 3, \tag{4.13c}$$

$$(\mathbf{MW}''' + 3\mathbf{M}'\mathbf{W}'' + 3\mathbf{M}''\mathbf{W}' + \mathbf{M}'''\mathbf{W})|_0 = \mathbf{0} \quad \text{if } p = 4. \tag{4.13d}$$

Let $\boldsymbol{\eta}(\mu)$ be a column of the matrix function $\mathbf{W}(\mu)$, i.e., $\boldsymbol{\eta}(\mu) = \mathbf{W}(\mu)\boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is a column of $\mathbf{I}_3$. Eq. (4.13a–d) imply that the premises of Theorem 4 are satisfied for every n such that $n \leqslant p-1$. Hence $\mathbf{JW}\boldsymbol{\rho}|_0, (\mathbf{JW}\boldsymbol{\rho})'|_0, \ldots, (\mathbf{JW}\boldsymbol{\rho})^{(n)}|_0$ are all eigenvectors provided that they are not null. For a normal eigenvalue $\mu_0, \mathbf{W}(\mu_0)$ is a rank one symmetric matrix so that, according to Eq. (3.3), it has at least one nonzero diagonal element W_{kk}. Then the kth column of $\mathbf{W}(\mu_0)$ is not a null vector, and the following theorem is established.

Theorem 5 (Eigensolutions of a normal eigenvalue). *If μ_0 is a normal eigenvalue of multiplicity p, then $\mathbf{W}(\mu_0)$ has a nonzero diagonal element $\boldsymbol{\rho}^{\mathrm{T}}\mathbf{W}(\mu_0)\boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is a column vector of $\mathbf{I}_3$. A set of p eigenvectors of the orders increasing from 0 to $p-1$ is given as follows along with the corresponding eigensolutions*:

$$\boldsymbol{\xi}^{[j]} = (\mathbf{JW})^{(j)}\boldsymbol{\rho}|_0 = \{\mathbf{JW}^{(j)} + (j,1)\mathbf{J}'\mathbf{W}^{(j-1)} + (j,2)\mathbf{J}''\mathbf{W}^{(j-2)} + (j,3)\mathbf{J}'''\mathbf{W}^{(j-3)}\}\boldsymbol{\rho}|_0, \quad (j = 0, \ldots, p-1) \tag{4.14a}$$

$$\boldsymbol{\chi}^{[N]} = (f_N\mathbf{JW})^{(N)}\boldsymbol{\rho}|_0 = \sum_{0\leqslant j\leqslant N}(N,j)(\mathrm{d}^j f_N/\mathrm{d}\mu^j)|_0\boldsymbol{\xi}^{[N-j]} \quad (N = 0, \ldots, p-1), \tag{4.14b}$$

where it is understood that $\mathbf{W}^{(j)} = \mathbf{0}$ if j is negative.

Eqs. (4.5) and (4.14a) imply an important expression for the binary product of any two eigenvectors of arbitrary orders ($0 \leqslant p, q \leqslant 3$) that share a common eigenvalue:

$$\begin{aligned}[\![\boldsymbol{\xi}^{[p]}(\mu), \boldsymbol{\xi}^{[q]}(\mu)]\!] &= \sum_{0\leqslant k\leqslant p}\sum_{0\leqslant l\leqslant q}(p,k)(q,l)\{k!l!/(k+l+1)!\}(\boldsymbol{\eta}^{(p-k)})^{\mathrm{T}}\mathbf{M}^{(k+l+1)}\boldsymbol{\eta}^{(q-l)} \\ &= p!q!\sum_{q+1\leqslant s\leqslant p+q+1}\{s!(p+q+1-s)!\}^{-1}(\boldsymbol{\eta}^{(p+q+1-s)})^{\mathrm{T}}\sum_{s-q\leqslant m\leqslant s}(s,m)\mathbf{M}^{(m)}\boldsymbol{\eta}^{(s-m)}.\end{aligned} \tag{4.15}$$

Eqs. (4.1)–(4.4) and (4.15) are formally identical to the corresponding relations for the eigenvectors of coupled anisotropic laminated plates (Yin, 2003a), but the physical meaning of the variables and of their relations are very different in the two cases. Although the two key matrix functions $\mathbf{J}(\mu)$ and $\mathbf{M}(\mu)$ are different for the two theories, they nonetheless satisfy the same set of formal identities (4.1)–(4.5) and (4.15). Some important results concerning the algebraic structure and properties of the solution spaces follow directly from these identities, regardless of the specific forms of material matrices and constitutive relations. This is the underlying reason of the formal similarity of the different theories in many aspects. However, the material functions $\mathbf{J}(\mu)$ and $\mathbf{M}(\mu)$ of the two theories have different dimensions and contain elements of different polynomial degrees. Such differences do affect the number and variety of cases in each theory, and the forms of the general solution in the different cases.

4.4. Eigenspace of a normal eigenvalue

The eigenvectors of the various orders associated with a common multiple eigenvalue are generally not orthogonal. For a normal eigenvalue μ_0, the eigenvectors are given by Eq. (4.14a), where $\boldsymbol{\rho}$ is a column of $\mathbf{I}_3$ such that

$$W \equiv \boldsymbol{\rho}^{\mathrm{T}}\mathbf{W}(\mu_0)\boldsymbol{\rho} \neq 0. \tag{4.16}$$

Using Eqs. (3.1) and (3.2), Eq. (4.15) becomes

$$\begin{aligned}
&[\![\boldsymbol{\xi}^{[p]}(\mu), \boldsymbol{\xi}^{[q]}(\mu)]\!] \\
&\quad = (W\delta)^{(p+q+1)}p!q!/(p+q+1)! \\
&\quad = \{\delta^{(p+q+1)}W + (p+q+1)\delta^{(p+q)}W' + (p+q+1,2)\delta^{(p+q-1)}W'' \\
&\qquad + (p+q+1,3)\delta^{(p+q-2)}W'''\}p!q!/(p+q+1)!
\end{aligned} \tag{4.17a}$$

where any negative-order derivative that appears is taken to be zero, and

$$W^{(k)} \equiv \boldsymbol{\rho}^{\mathrm{T}}\mathbf{W}^{(k)}(\mu_0)\boldsymbol{\rho} \quad (1 \leqslant k \leqslant 3). \tag{4.17b}$$

Let the p eigenvectors of the normal eigenvalue μ_0 of multiplicity p be arranged in the increasing order as the columns of a $8\times p$ matrix $\mathbf{X}_p$, and define the pseudometric of the eigenspace $\boldsymbol{\omega}_{[Np]} = [\![\mathbf{X}_p, \mathbf{X}_p]\!]$. Then, in view of Eq. (4.17a), the following expressions apply for $p = 1, 2, 3$ and 4, respectively,

$$\boldsymbol{\omega}_{[N1]} = [\delta' W], \quad \boldsymbol{\omega}_{[N2]} = \begin{bmatrix} 0 & \delta'' W/2 \\ \delta'' W/2 & \delta''' W/6 + \delta'' W'/2 \end{bmatrix} \tag{4.18a,b}$$

$$\boldsymbol{\omega}_{[N3]} = \begin{bmatrix} 0 & 0 & \delta''' W/3 \\ 0 & \delta''' W/6 & \delta^{(4)} W/12 + \delta''' W'/3 \\ \delta''' W/3 & \delta^{(4)} W/2 + \delta''' W'/3 & \delta^{(5)} W/30 + \delta^{(4)} W'/6 + \delta''' W''/3 \end{bmatrix} \tag{4.18c}$$

$$\boldsymbol{\omega}_{[N4]} = \begin{bmatrix} 0 & 0 & 0 & \delta^{(4)} W/4 \\ 0 & 0 & \delta^{(4)} W/12 & \delta^{(5)} W/20 + \delta^{(4)} W'/4 \\ 0 & \delta^{(4)} W/12 & \delta^{(5)} W/30 + \delta^{(4)} W'/6 & \omega_{34} \\ \delta^{(4)} W/4 & \delta^{(5)} W/20 + \delta^{(4)} W'/4 & \omega_{34} & \omega_{44} \end{bmatrix} \tag{4.18d}$$

where

$$\omega_{34} \equiv \{\delta^{(6)} W + 6\delta^{(5)} W' + 15\delta^{(4)} W''\}/60,$$

$$\omega_{44} \equiv \{\delta^{(7)} W + 7\delta^{(5)} W' + 21\delta^{(5)} W'' + 35\delta^{(4)} W'''\}/140.$$

Hence, for a normal eigenvalue, the matrices $\boldsymbol{\omega}_{[N1]}$, $\boldsymbol{\omega}_{[N2]}$, $\boldsymbol{\omega}_{[N3]}$ and $\boldsymbol{\omega}_{[N4]}$ are formally identical to the corresponding pseudometrics of coupled anisotropic laminated plates. The four matrices are all invertible, and their inverse matrices may be given in closed analytic forms (identical to Eqs. (A.1)–(A.4) of Yin, 2003a). They are called pseudometrics because the matrix $\boldsymbol{\omega}$ resulting from taking the binary products of the eigenvectors of the various orders is not real-valued, let alone positive definite. Although orthogonality of vectors has been defined in terms of the binary product, length and angles are undefined in the complex vector space. Any vanishing diagonal element in the matrix $\boldsymbol{\omega}$, such as in Eq. (4.18b–d), implies that the corresponding column vector of $\mathbf{X}_p$ has a vanishing binary product with itself, and therefore cannot be normalized.

One of the reasons that the pseudometric plays an important role is given by the following theorem, which is valid irrespective of the type of eigenvalue:

Theorem 6 (Independence of eigenvectors). *A set of (eight-dimensional) vectors is linearly independent if the pseudometric formed by these vectors is a nonsingular matrix.*

Proof. Let $\mathbf{X}_p$ be an $8 \times p$ matrix. If $\boldsymbol{\omega} \equiv \mathbf{X}_p^{\mathrm{T}}\mathbf{II}\mathbf{X}_p$ is nonsingular, then for every nontrivial p-dimensional vector $\mathbf{f}$, the image vector $\mathbf{X}_p^{\mathrm{T}}\mathbf{II}\mathbf{X}_p\mathbf{f}$ is nontrivial. Then $\mathbf{X}_p\mathbf{f}$ must be nontrivial. This implies that a linear combination of the p columns of $\mathbf{X}_p$ vanishes only if all coefficients of the combination vanish, and hence the columns are linearly independent. □

Only eigenvalues, but not eigenvectors and eigensolutions, are uniquely determined by the material. Expressions different from, but equivalent to (4.14a,b), may be given for the eigensolutions of a normal eigenvalue. If the eigenvectors $(\mathbf{JW})^{(k)}\boldsymbol{\rho}|_0$ are replaced by $(\mathbf{JW}/\sqrt{W})^{(k)}\boldsymbol{\rho}|_0$, then, instead of Eqs. (4.17) and (4.18), one obtains a more concise form of the pseudometric that is independent of W and its derivatives, and involves the derivatives of δ only

$$\boldsymbol{\omega}_{ij} = [\![\boldsymbol{\xi}^{[i-1]}(\mu), \boldsymbol{\xi}^{[j-1]}(\mu)]\!] = \delta^{(i+j-1)}(i-1)!(j-1)!/(i+j-1)! \tag{4.19}$$

In the literature on anisotropic elasticity, the eigenvectors are sometimes normalized so as to have unit binary products with themselves. Such normalization serves no useful purpose, often leads to more complicated results (in contrast to the simplification shown by Eq. (4.19) which uses eigenvectors whose binary product with itself is not unity), and is not even possible for eigenvectors of abnormal and superabnormal eigenvalues that have vanishing binary products with themselves.

5. Higher-order eigenvectors of abnormal eigenvalues

5.1. Eigenspace of a double abnormal eigenvalue

For an abnormal eigenvalue μ_0 of multiplicity 2, the matrix $M(\mu_0)$ has the form of Eq. (3.7b) with a nonvanishing kth diagonal element v_k. Two independent zeroth-order eigenvectors are $\mathbf{J}\boldsymbol{\eta}_1|_0$ and $\mathbf{J}\boldsymbol{\eta}_2|_0$, where $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are given by Eq. (3.10a–c), depending on the integer k. Taking the example $k = 2$, the matrix function $\mathbf{U}_k(\mu)$ has the expression

$$\mathbf{U}_2(\mu) = \begin{bmatrix} M_{22}(\mu) & M_{12}(\mu) & 0 \\ -M_{12}(\mu) & M_{22}(\mu) & -M_{23}(\mu) \\ 0 & M_{23}(\mu) & M_{22}(\mu) \end{bmatrix}. \tag{5.1}$$

Then

$$\mathbf{M}(\mu)\mathbf{U}_2(\mu)(\mathbf{I}_3 - \boldsymbol{\rho}_k\boldsymbol{\rho}_k^{\mathrm{T}}) = \begin{bmatrix} W_{33}(\mu) & 0 & -W_{13}(\mu) \\ 0 & 0 & 0 \\ -W_{13}(\mu) & 0 & W_{11}(\mu) \end{bmatrix}. \tag{5.2}$$

The expression vanishes at $\mu = \mu_0$ since $\mathbf{W}(\mu_0) = 0$. Differentiating this identity, evaluating at $\mu = \mu_0$, premultiplying the result by $\mathbf{W}'(\mu_0)$, and using Eqs. (3.6a,b) and (5.2), one obtains,

$$\begin{bmatrix} W'_{11}(\mu_0) & W'_{12}(\mu_0) & W'_{13}(\mu_0) \\ W'_{12}(\mu_0) & W'_{22}(\mu_0) & W'_{23}(\mu_0) \\ W'_{13}(\mu_0) & W'_{23}(\mu_0) & W'_{33}(\mu_0) \end{bmatrix} \begin{bmatrix} W'_{33}(\mu_0) & 0 & -W'_{13}(\mu_0) \\ 0 & 0 & 0 \\ -W'_{13}(\mu_0) & 0 & W'_{11}(\mu_0) \end{bmatrix}$$
$$= \mathbf{W}'\mathbf{M}'\mathbf{U}_2(\mathbf{I}_3 - \boldsymbol{\rho}_k\boldsymbol{\rho}_k^{\mathrm{T}})|_0 + \mathbf{W}'\mathbf{M}\mathbf{U}'_2(\mathbf{I}_3 - \boldsymbol{\rho}_k\boldsymbol{\rho}_k^{\mathrm{T}})|_0 = (\delta''/2)\mathbf{U}_2(\mathbf{I}_3 - \boldsymbol{\rho}_k\boldsymbol{\rho}_k^{\mathrm{T}})|_0,$$

where $\delta''(\mu_0)$ does not vanish because μ_0 is a double eigenvalue. After deleting the kth row and the kth column, the preceding equalities reduce to the following expression evaluated at μ_0

$$\begin{bmatrix} W'_{11} & W'_{13} \\ W'_{13} & W'_{33} \end{bmatrix} \begin{bmatrix} W'_{33} & -W'_{13} \\ -W'_{13} & W'_{11} \end{bmatrix} = (\delta''/2)M_{22}\mathbf{I}_2 \tag{5.3}$$

Since $[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2]$ is the 3×2 matrix obtained by deleting the kth column of $\mathbf{U}_2$, the pseudometric

$$\boldsymbol{\omega} = [\![[\mathbf{J}\boldsymbol{\eta}_1, \mathbf{J}\boldsymbol{\eta}_2], [\mathbf{J}\boldsymbol{\eta}_1, \mathbf{J}\boldsymbol{\eta}_2]]\!] = [\boldsymbol{\eta}_1, \boldsymbol{\eta}_2]^{\mathrm{T}}[\![\mathbf{J}, \mathbf{J}]\!][\boldsymbol{\eta}_1, \boldsymbol{\eta}_2] = [\boldsymbol{\eta}_1, \boldsymbol{\eta}_2]^{\mathrm{T}}\mathbf{M}'(\mu_0)[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2]$$

is obtained from the 3×3 matrix $(\mathbf{U}_k^{\mathrm{T}}\mathbf{M}'\mathbf{U}_k)|_0$ by eliminating the kth row and k th column. The latter may be obtained by using Eq. (5.1) and the derivative of (5.2). One has

$$\boldsymbol{\omega}_{[A2]} = M_{22}\begin{bmatrix} W'_{33} & -W'_{13} \\ -W'_{13} & W'_{11} \end{bmatrix} \quad \boldsymbol{\omega}^{-1}_{[A2]} = 2/(\delta'' M_{22}^2)\begin{bmatrix} W'_{11} & W'_{13} \\ W'_{13} & W'_{33} \end{bmatrix}, \tag{5.4a,b}$$

where (5.4b) follows from Eq. (5.3). For the cases $k = 1$ and $k = 3$, the results for $\boldsymbol{\omega}$ and $\boldsymbol{\omega}^{-1}$ may be obtained by permuting the indices. They imply, in particular, that $W'(\mu_0)$ is not a null matrix.

Since $\boldsymbol{\omega}$ is a symmetric matrix, a 2×2 nonsingular matrix $\boldsymbol{\tau}$ can always be found such that $\boldsymbol{\Delta} \equiv \boldsymbol{\tau}^{\mathrm{T}}\boldsymbol{\omega}\boldsymbol{\tau}$ is a diagonal matrix. Then $\boldsymbol{\Delta}$ is the pseudometric referred to another set of two eigenvectors given by the two columns of $\mathbf{J}[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2]\boldsymbol{\tau}|_0$, because

$$[\![\mathbf{J}[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2]\boldsymbol{\tau}, \mathbf{J}[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2]\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^{\mathrm{T}}[\![\mathbf{J}[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2], \mathbf{J}[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2]]\!]\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{T}}\boldsymbol{\omega}\boldsymbol{\tau}. \tag{5.5}$$

Theorem 7 (Eigensolutions of a double abnormal eigenvalue). *For a double abnormal eigenvalue μ_0, $\mathbf{M}(\mu_0)$ has a nonvanishing diagonal element $M_{kk}(\mu_0)$. Let $\boldsymbol{\xi}_1 = \mathbf{J}\boldsymbol{\eta}_1|_0$ and $\boldsymbol{\xi}_2 = \mathbf{J}\boldsymbol{\eta}_2|_0$, where $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are the two remaining columns of the matrix function $\mathbf{U}_k(\mu)$ of Eq. (3.8) after having the kth column deleted. Then $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are zeroth-order eigenvectors. They determine a pseudometric $\boldsymbol{\omega}$ such that $(\delta'' M_{kk}^2)\boldsymbol{\omega}^{-1}$ is the 2×2 matrix obtained from $\mathbf{W}'(\mu_0)$ by deleting the kth row and kth column. Furthermore, a nonsingular transformation $\boldsymbol{\tau}$ may be found so that the transformed zeroth-order eigenvectors $\boldsymbol{\xi}'_1 = \boldsymbol{\xi}_1\boldsymbol{\tau}$ and $\boldsymbol{\xi}'_2 = \boldsymbol{\xi}_2\boldsymbol{\tau}$, yield a pseudometric $\boldsymbol{\Delta} = \boldsymbol{\tau}^T\boldsymbol{\omega}\boldsymbol{\tau}$ which is a diagonal matrix.*

Notice that the present results for a double abnormal eigenvalue are quite different from the results for a double abnormal eigenvalue in 2-D elasticity and in the coupled anisotropic plate theory. In the latter results, the two zeroth-order eigenvectors are simply given by the two columns of $\mathbf{J}(\mu_0)$ [there $\mathbf{J}(\mu_0)$ has the dimension 6×2 or 8×2 in contrast to the present $J(\mu_0)$ that has the dimension 8×3].

5.2. Abnormal-α eigenvalues

If $p \geqslant 3$ for an abnormal eigenvalue, or if $p = 4$ for a superabnormal eigenvalue, then high-order eigensolutions are needed to make up for the deficiency in the independent eigensolutions. We first consider abnormal eigenvalues. With a null matrix $\mathbf{W}(\mu_0)$, Eq. (4.13b–d) reduce to the following at $\mu = \mu_0$:

$$\mathbf{M}\mathbf{W}'|_0 = \mathbf{0}, \quad (2\mathbf{M}'\mathbf{W}' + \mathbf{M}\mathbf{W}'')|_0 = \mathbf{0} \quad \text{if } p \geqslant 3. \tag{5.6a,b}$$

$$(3\mathbf{M}''\mathbf{W}' + 3\mathbf{M}'\mathbf{W}'' + \mathbf{M}\mathbf{W}''')|_0 = \mathbf{0} \quad \text{if } p = 4. \tag{5.6c}$$

Taking $\boldsymbol{\eta}(\mu)$ to be a column of $\mathbf{W}(\mu)$, then the premises of Theorem 4 are satisfied but $\boldsymbol{\eta}(\mu_0)$ and $\mathbf{J}\boldsymbol{\eta}|_0$ are null vectors. Hence if $(\mathbf{J}\boldsymbol{\eta})^{(k)}|_0$ is not null, it is an eigenvector of order lower than k.

For an abnormal-α eigenvalue of multiplicity p, $\mathbf{W}'(\mu_0)$ has a nontrivial column $\mathbf{W}'(\mu_0)\boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is the kth column of $\mathbf{I}_3$. Then $(\mathbf{JW})^{(j)}\boldsymbol{\rho}|_0$ is an eigenvector of the order $j-1$ ($j = 1,2,\ldots,p-1$). Hence one has the following zeroth-order and first-order eigensolutions for $p \geqslant 3$:

$$\boldsymbol{\xi}_1 = (\mathbf{JW})'\boldsymbol{\rho}|_0 = \mathbf{JW}'\boldsymbol{\rho}|_0, \quad \boldsymbol{\chi}_1 = (f\mathbf{JW})'\boldsymbol{\rho}|_0 = f(\mu_0)\boldsymbol{\xi}_1, \tag{5.7a,b}$$

$$\boldsymbol{\xi}_2 = (\mathbf{JW})''\boldsymbol{\rho}|_0 = (\mathbf{JW}'' + 2\mathbf{J}'\mathbf{W}')\boldsymbol{\rho}|_0, \tag{5.8a}$$

$$\boldsymbol{\chi}_2 = (f\mathbf{JW})''\boldsymbol{\rho}|_0 = f(\mu_0)\boldsymbol{\xi}_2 + 2f'(\mu_0)\boldsymbol{\xi}_1. \tag{5.8b}$$

If $p = 4$, a second-order eigensolution is given by

$$\boldsymbol{\xi}_3 = (\mathbf{JW})'''\boldsymbol{\rho}|_0 = (\mathbf{JW}''' + 3\mathbf{J}'\mathbf{W}'' + 3\mathbf{J}''\mathbf{W}')\boldsymbol{\rho}|_0, \tag{5.9a}$$

$$\boldsymbol{\chi}_3 = (f\mathbf{JW})'''\boldsymbol{\rho}|_0 = f(\mu_0)\boldsymbol{\xi}_3 + 3f'(\mu_0)\boldsymbol{\xi}_2 + 3f''(\mu_0)\boldsymbol{\xi}_1. \tag{5.9b}$$

Eqs. (5.8)–(5.10) give $p-1$ eigenvectors for an abnormal-α eigenvalue. An additional (zeroth-order) eigenvector remains to be found, and has to be obtained in different ways, depending on the matrices $\mathbf{M}(\mu_0)$ and $\mathbf{W}'(\mu_0)$. It will be obtained using Eq. (3.10) and the following two lemmas, whose proof is given in Appendix A.

Lemma 1. *Consider the symmetric matrix function* $\mathbf{M}(\mu)$ *whose elements are given by Eq.* (2.20). *If* $\mathbf{M}(\mu_0)$ *is of rank* 1 *and* $p \geqslant 3$, *then* $M_{22}(\mu_0)$ *and* $M_{33}(\mu_0)$ *cannot both vanish.*

Lemma 2. *For every abnormal eigenvalue* μ_0 *of multiplicity* $p \geqslant 3$ *such that* $\mathbf{W}'(\mu_0)$ *is not null, there exist two distinct indices i and j such that* $M_{ii}(\mu_0) \neq 0$ *and* $W'_{jj}(\mu_0) \neq 0$.

Lemmas 1 and 2 imply that there are only four possible cases of abnormal-α eigenvalue:

(i) $M_{22}(\mu_0) \neq 0$ and $W'_{11}(\mu_0) \neq 0$;
(ii) $M_{22}(\mu_0) \neq 0$ and $W'_{33}(\mu_0) \neq 0$;
(iii) $M_{33}(\mu_0) \neq 0$ and $W'_{11}(\mu_0) \neq 0$;
(iv) $M_{33}(\mu_0) \neq 0$ and $W'_{22}(\mu_0) \neq 0$.

The eigenvectors for these four subcases are given by the following theorem.

Theorem 8 (Eigenvectors of an abnormal-α eigenvalue). *For an abnormal-α eigenvalue, one zeroth-order eigenvector* $\boldsymbol{\xi}_1$ *is given as follows for the cases (i)–(iv), respectively*

$$\mathbf{J}\{0, -M_{23}, M_{22}\}|_0, \mathbf{J}\{M_{22}, -M_{12}, 0\}|_0, \mathbf{J}\{0, M_{33}, -M_{23}\}|_0, \mathbf{J}\{M_{33}, 0, -M_{13}\}|_0. \tag{5.10a}$$

Additional eigenvectors of the zeroth and higher orders are given by

$$\boldsymbol{\xi}_2 = \mathbf{JW}'\boldsymbol{\rho}|_0, \ldots, \boldsymbol{\xi}_p = (\mathbf{JW})^{(p-1)}\boldsymbol{\rho}|_0. \tag{5.10b}$$

The complete set consists of p independent eigenvectors.

These eigenvectors determine the following pseudometrics for $p = 3$ and 4, respectively, and their invertibility implies the independence of the eigenvectors:

$$\boldsymbol{\omega} = \begin{bmatrix} M_{jj}W'_{kk} & 0 & 0 \\ 0 & 0 & \delta''' W'_{kk}/3 \\ 0 & \delta''' W'_{kk}/3 & \delta''' W''_{kk}/3 + \delta^{(4)} W'_{kk}/6 \end{bmatrix}, \tag{5.11a}$$

$$\boldsymbol{\omega}_{[N4]} = \begin{bmatrix} M_{jj}W'_{kk} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta^{(4)} W'_{kk}/4 \\ 0 & 0 & \delta^{(4)} W'_{kk}/6 & \delta^{(4)} W''_{kk}/4 + \delta^{(5)} W'_{kk}/10 \\ 0 & \delta^{(4)} W'_{kk}/4 & \delta^{(4)} W''_{kk}/4 + \delta^{(5)} W'_{kk}/10 & \delta^{(4)} W'''_{kk}/4 + 3\delta^{(5)} W''_{kk}/20 + \delta^{(6)} W'_{kk}/20 \end{bmatrix}, \tag{5.11b}$$

where M_{jj} and W'_{kk} are the nonvanishing diagonal elements of $\mathbf{M}(\mu_0)$ and $\mathbf{W}'(\mu_0)$ that appear in the definition of each cases (i)–(iv).

5.3. Abnormal-β eigenvalues

Next consider an abnormal-β eigenvalue, for which the matrix $\mathbf{W}'(\mu_0)$ is null. None of the elements in the first row and first column of $\mathbf{W}''(\mu_0)$ can vanish, because the corresponding elements of $\mathbf{W}(\mu)$ are quartic and quintic functions with real coefficients, so that they cannot vanish at $\mu = \mu_0$ along with their first and second derivatives. We let $\boldsymbol{\eta}(\mu) = \mathbf{W}(\mu)\boldsymbol{\rho}_1$, where $\boldsymbol{\rho}_1$ is the first column of $\mathbf{I}_3$. Then, with $\mathbf{W}'(\mu_0)$ dropped and with $\boldsymbol{\rho}$ replaced by $\boldsymbol{\rho}_1$, the zeroth-order eigenvector of Eq. (5.7) becomes the null vector, while the first- and second-order eigensolutions of Eqs. (5.8) and (5.9) change into zeroth-and first-order eigensolutions, respectively. This yields $p-2$ eigensolutions, so that two additional eigensolutions remain to be found, one zeroth-order and another first-order.

The premises of Lemma 1 are satisfied by an abnormal-β eigenvalue of multiplicity 3 or 4. One needs only consider the case $M_{22}(\mu_0) \neq 0$, since the other case $M_{33}(\mu_0) \neq 0$ is similar. We choose $\boldsymbol{\eta}(\mu) = \{0, -M_{23}(\mu), M_{22}(\mu)\}^{\mathrm{T}}$, in accordance with $\boldsymbol{\eta}_2$ in Eq. (3.10b). Then $\mathbf{M}\boldsymbol{\eta} = \{-W_{13}(\mu), 0, W_{11}(\mu)\}^{\mathrm{T}}$, so that $\mathbf{M}\boldsymbol{\eta}|_0 = (\mathbf{M}\boldsymbol{\eta})'|_0 = \mathbf{0}$ follows from $\mathbf{W}(\mu_0) = \mathbf{W}'(\mu_0) = \mathbf{0}$. Thus $\mathbf{J}\boldsymbol{\eta}|_0$ is a zeroth-order eigenvector and $(\mathbf{J}\boldsymbol{\eta})'|_0$ is of the first-order. These eigenvectors together with the previous $p-2$ eigenvectors based on $\boldsymbol{\eta} = \mathbf{W}\boldsymbol{\rho}_1$ form a complete set of p eigenvectors associated with μ_0:

Theorem 9 (Eigensolutions of an abnormal-β eigenvalue). *For an abnormal-β eigenvalue μ_0 (of multiplicity $p \geqslant 3$), one zeroth-order eigensolution $\boldsymbol{\chi}_1$ and one first-order eigensolution $\boldsymbol{\chi}_2$ are given as follows provided that* $M_{22}(\mu_0) \neq 0$

$$\boldsymbol{\xi}_1 = \mathbf{J}\{0, -M_{23}, M_{22}\}^{\mathrm{T}}|_0, \quad \boldsymbol{\chi}_1 = f(x + \mu_0 y)\boldsymbol{\xi}_1, \tag{5.12a,b}$$

$$\boldsymbol{\xi}_2 = \mathrm{d}/\mathrm{d}\mu(\mathbf{J}\{0, -M_{23}, M_{22}\}^{\mathrm{T}})|_0, \quad \boldsymbol{\chi}_2 = f(x + \mu_0 y)\boldsymbol{\xi}_2 + f'(x + \mu_0 y)\boldsymbol{\xi}_1. \tag{5.13a,b}$$

If $M_{22}(\mu_0) = 0$, then the vector function $\{0, -M_{23}, M_{22}\}^T$ in the preceding expressions should be replaced by $\{0, M_{33}, -M_{23}\}^T$. A second zeroth-order eigensolution is given by

$$\boldsymbol{\xi}_3 = (\mathbf{J}\mathbf{W})''\boldsymbol{\rho}_1|_0 = \mathbf{J}\mathbf{W}''\boldsymbol{\rho}_1|_0, \quad \boldsymbol{\chi}_3 = (f\mathbf{J}\mathbf{W})''\boldsymbol{\rho}_1|_0 = f(\mu_0)\boldsymbol{\xi}_1. \tag{5.14a,b}$$

If $p = 4$, then one has another first-order eigensolution:

$$\boldsymbol{\xi}_4 = (\mathbf{J}\mathbf{W})'''\boldsymbol{\rho}_1|_0 = (\mathbf{J}\mathbf{W}''' + 3\mathbf{J}'\mathbf{W}'')\boldsymbol{\rho}_1|_0, \tag{5.15a}$$

$$\boldsymbol{\chi}_4 = (f\mathbf{J}\mathbf{W})'''\boldsymbol{\rho}_1|_0 = f(x + \mu_0 y)\boldsymbol{\xi}_4 + 3f'(x + \mu_0 y)\mathbf{J}\mathbf{W}''\boldsymbol{\rho}_1|_0. \tag{5.15b}$$

The preceding eigenvectors yield the pseudometric:

$$\boldsymbol{\omega} = \langle \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \rangle, \tag{5.16a}$$

where $\langle \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \rangle$ denotes a block-diagonal matrix composed of two diagonal blocks, with

$$\boldsymbol{\omega}_1 = \begin{bmatrix} 0 & M_{kk}W''_{11}/2 \\ M_{kk}W''_{11}/2 & (M_{kk}W'''_{11} + 3M'_{kk}W''_{11})/6 \end{bmatrix} \tag{5.16b}$$

and $\boldsymbol{\omega}_2$ is given by the following 1×1 or 2×2 matrix for $p = 3$ and 4, respectively

$$[\delta''' W_{11}], \begin{bmatrix} 0 & \delta^{(4)} W''_{11}/4 \\ \delta^{(4)} W''_{11}/4 & \delta^{(4)} W'''_{11}/4 + (3/20)\delta^{(5)} W''_{11} \end{bmatrix} \tag{5.16c,d}$$

5.4. Eigenspace of a superabnormal eigenvalue

For a superabnormal eigenvalue μ_0, the matrix $\mathbf{M}(\mu_0)$ is null so that $\mathbf{J}(\mu_0)\boldsymbol{\rho}$ is an eigenvector for whatever $\boldsymbol{\eta}$. Choosing $\boldsymbol{\eta}$ to be the three columns of $\mathbf{I}_3$, successively, one obtains three independent zeroth-order eigenvectors given by the three columns of $\mathbf{J}(\mu_0)$. Multiplication of each eigenvector by an analytic function results in a set of three independent zeroth-order eigensolutions. For $p = 3$, the three eigenvectors yield the pseudometric

$$\boldsymbol{\omega} = [\![\mathbf{J}(\mu_0), \mathbf{J}(\mu_0)]\!] = \mathbf{M}'(\mu_0). \tag{5.17}$$

Since $\mathbf{M}(\mu_0)$ is null, the adjoint matrix $\mathbf{W}(\mu_0)$ and its derivative $\mathbf{W}'(\mu_0)$ are both null. Eq. (3.2) with $n = 3$ reduces to $3\mathbf{M}'\mathbf{W}'' = \delta'''\mathbf{I}_3$. Hence $\mathbf{M}'$ is invertible and

$$\boldsymbol{\omega}^{-1} = \{3/\delta'''(\mu_0)\}\mathbf{W}''(\mu_0). \tag{5.18}$$

If $p = 4$, we choose the first two zeroth-order eigensolutions as follows:

$$\boldsymbol{\xi}_1 = \mathbf{J}(\mu_0)\boldsymbol{\rho}_2, \quad \boldsymbol{\xi}_2 = \mathbf{J}(\mu_0)\boldsymbol{\rho}_3, \quad \boldsymbol{\chi}_1 = f(x + \mu_0 y)\boldsymbol{\xi}_1, \quad \boldsymbol{\chi}_2 = f(x + \mu_0 y)\boldsymbol{\xi}_2. \tag{5.19}$$

The third and fourth eigensolutions are chosen to be identical to $\boldsymbol{\chi}_3$ and $\boldsymbol{\chi}_4$ for the case of a quadruple abnormal-β eigenvalue, given by Eqs. (5.14a,b) and (5.15a,b), respectively. Then $\boldsymbol{\omega} = \langle \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \rangle$, where $\boldsymbol{\omega}_2$ is given by Eq. (5.16d), and $\boldsymbol{\omega}_1$ is the 2×2 submatrix at the lower right corner of $\mathbf{M}(\mu_0)$. Notice that $\boldsymbol{\omega}$ is nonsingular

$$\mathrm{Det}[\boldsymbol{\omega}] = -(W_{11}\delta''''/4)^2(M'_{22}M'_{22} - M'^2_{23})|_0 = -(W_{11}\delta''''/4)^2(\mu_0 - \bar{\mu}_0)^2(\beta_{55}\alpha_{11} + \gamma_{15}^2) > 0.$$

6. Concluding remarks

The key relation in 2-D piezoelectricity is Eq. (2.17). When the stress components and the electric displacement are considered as the primary unknown variables, this eigenrelation directly implies Eq. (2.18), from which both the eigenvalues and the reduced eigenvector $\boldsymbol{\eta}$ may be solved in terms of the elements $M_{ij}(\mu_0)$ of Eq. (2.20). All elements are polynomial functions of μ. The resulting eigenvectors are also polynomial functions. The present choice of the primary and secondary unknown variables, and the adoption of $[\varpi]$ as the constitutive matrix, are in agreement with the formalism used by Sosa (1991) for special transversely isotropic materials. Due to uncoupling of antiplane mode from both in-plane mode and the electric field in that special case, the dimension of eigenvectors and the degree of the characteristic equation both reduce from eight to six.

In most works on piezoelectricity (Tiersten, 1969; Wang and Zheng, 1995; Sosa and Castro, 1994; Bisegna and Maceri, 1996; Ding et al., 1996; Heyliger, 1997; Vel and Batra, 2000; Shodja and Kamali, 2003), the primary unknown variables are taken to be the displacements and the electric field. The constitutive matrix used in such theories is the *inverse matrix* of $[\varpi]$, since the roles of the primary and secondary unknown variables are reversed. The inverse formalism is an extension of the Eshelby–Stroh formalism of 2-D elasticity, with the electric field potential included as an additional primary variable. The key relations of this inverse formalism also follow directly from the basic eigenrelation of Eq. (2.17), but in a different manner. Premultiplying Eq. (2.17) by $\mathbf{\Theta}^{\mathrm{T}}[\varpi]^{-1}$, one obtains

$$\mathbf{\Theta}^{\mathrm{T}}[\varpi]^{-1}\mathbf{\Theta}\{\xi_5,\xi_6,\xi_7,\xi_8\}^{\mathrm{T}} = 0. \tag{6.1}$$

Hence the eigenvalues are determined by the characteristic equation

$$\mathrm{Det}[\mathbf{\Theta}^{\mathrm{T}}[\varpi]^{-1}\mathbf{\Theta}] = 0, \tag{6.2}$$

which is certainly equivalent to Eq. (2.21), because both characteristic equations issue from the same eigenrelation (2.17), which may be rewritten in two alternative forms:

$$[[\varpi]\mathbf{\Psi},\mathbf{\Theta}]\{\xi_2,\xi_3,\xi_4,\xi_5,\xi_6,\xi_7,\xi_8\}^{\mathrm{T}} = \mathbf{0}, \tag{6.3a}$$

$$[\mathbf{\Psi},[\varpi]^{-1}\mathbf{\Theta}]\{\xi_2,\xi_3,\xi_4,\xi_5,\xi_6,\xi_7,\xi_8\}^{\mathrm{T}} = \mathbf{0}. \tag{6.3b}$$

The first expression yields Eq. (2.18) after eliminating $\{\xi_5,\xi_6,\xi_7,\xi_8\}^{\mathrm{T}}$ whereas the second yields Eq. (6.1) after eliminating $\{\xi_2,\xi_3,\xi_4\}^{\mathrm{T}}$ Therefore, the eigenvalues of Eqs. (2.18) and (6.1) are both identical to those of Eq. (6.3), and hence cannot be mutually different. Furthermore, for each solution $\{\xi_2,\xi_3,\xi_4\}^{\mathrm{T}}$ of Eq. (2.18), the eigenrelation (2.17) determines a solution $\{\xi_5,\xi_6,\xi_7,\xi_8\}^{\mathrm{T}}$ of Eq. (6.1), and vice versa, as follows:

$$\{\xi_5,\xi_6,\xi_7,\xi_8\}^{\mathrm{T}} = \mathbf{P}_1(\mu_0)[\varpi]\mathbf{\Psi}(\mu_0)\{\xi_2,\xi_3,\xi_4\}^{\mathrm{T}}, \tag{6.4a}$$

$$\{\xi_2,\xi_3,\xi_4\}^{\mathrm{T}} = \mathbf{P}_2(\mu_0)[\varpi]^{-1}\mathbf{\Theta}(\mu_0)\{\xi_5,\xi_6,\xi_7,\xi_8\}^{\mathrm{T}}, \tag{6.4b}$$

where

$$\mathbf{P}_1(\mu) \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{P}_2(\mu) \equiv \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Clearly, this simple proof of the equivalence of the two formalisms is also valid for 2-D anisotropic elasticity, which is a special case without piezoelectric coupling.

Eq. (6.3) clearly shows the dualism of the eigenrelation with respect to the two sets of variables $\{\xi_2,\xi_3,\xi_4\}$ and $\{\xi_5,\xi_6,\xi_7,\xi_8\}$. But there is an asymmetry in the dualism, because the existence of the Airy stress function implies that $\xi_1 = -\mu\xi_2$ and there is no corresponding relation for $\{\xi_5,\xi_6,\xi_7,\xi_8\}$. Although Eq. (6.4a,b) show that the present formalism and the inverse formalism are analytically equivalent, the resulting algebraic expressions of the eigenvectors and general solutions are entirely different, since all results of one formalism are expressed in terms of the elements of $[\varpi]$, whereas the other formalism involves the elements of $[\varpi]^{-1}$. The zeroth-order eigenvectors of the inverse formalism may be obtained from the explicit solutions $\{\xi_5,\xi_6,\xi_7,\xi_8\}^{\mathrm{T}}$ of Eq. (6.1). This explicit analytical solution, however, is far more cumbersome than the solution of Eq. (2.18), for the simple reason that the adjoint matrix of a 4×4 matrix $\mathbf{\Theta}^{\mathrm{T}}[\varpi]^{-1}\mathbf{\Theta}$is also 4×4, but each one of its elements is a 3×3 cofactor. Furthermore, the higher-order eigenvectors of degenerate materials are related to the zeroth-order ones by the derivative rule, and the complexity of expression grows dramatically when the μ-derivatives of the various order are taken from the lengthy analytical expressions of $\{\xi_5, \xi_6, \xi_7, \xi_8\}^T$ Indeed, none of the previous works using the inverse formalism has determined the explicit expressions of the eigenvectors of general 2-D piezoelectricity even for the nondegenerate case, and none can be used, in a practical sense, to obtain Green's functions and other important analytical solutions for the various degenerate cases.

There have been suggestions that a suitable choice of the formalism depends on the type of boundary conditions, e.g., whether the mechanical boundary conditions are kinematical or kinetic. However, the eight dimensional solution vector $\boldsymbol{\chi}$ obtained in the present analysis contains the stress potentials and the function ς as the first four elements, and u, v, w and ϕ as the last four elements. Because of the concurrent presence of the two groups of variables in $\boldsymbol{\chi}$, and because of the simple relations of Eq. (6.4a,b), the present form of solutions is indiscriminately applicable to all types of boundary conditions, whether they be kinematical, kinetic, or mixed, and whether the boundary conditions involve the electric field or the electric displacement vector. Hence the type of boundary conditions is not a factor affecting the suitability of a particular formalism. The proper choice is dictated only by the mathematical structure of the governing equations, i.e., Eq. (6.1) versus Eq. (2.18). Hence the excessive complexity associated with the inverse formalism is needlessly incurred, and brings no redeeming merit.

Appendix A

Proof of Lemma 1. Suppose the contrary were true, then $\mathbf{v} = \{v_1, 0, 0\}^T$ in Eq. (4.2) so that $M_{11}(\mu_0) = v_1$ is the only nonzero element of $\mathbf{M}(\mu_0)$. Hence, except for $M_{11}(\mu_0)$, all other elements of the matrix function $\mathbf{M}(\mu_0)$ contain the factor $\mu - \mu_0$. Being polynomial functions with real coefficients, they must also contain the factor $\mu - \bar{\mu}_0$. Then the three minors $W_{11}(\mu_0)$, $W_{12}(\mu_0)$ and $W_{13}(\mu_0)$ of Det[$\mathbf{M}(\mu_0)$] must contain the common factor $(\mu - \mu_0)^2(\mu - \bar{\mu}_0)^2$. Hence

$$\mathbf{M}(\mu_0) = \begin{bmatrix} v_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{W}'(\mu_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & W'_{22} & W'_{23} \\ 0 & W'_{23} & W'_{33} \end{bmatrix}. \tag{A.1a,b}$$

Furthermore, all elements of $\mathbf{M}'(\mu_0)$ do not vanish, except possibly $M'_{11}(\mu_0)$, because M_{12}, M_{13}, M_{22}, M_{23} and M_{33} are cubic or quadratic functions of μ and, therefore, cannot have the factor $(\mu - \mu_0)^2(\mu - \bar{\mu}_0)^2$. The preceding conclusions including Eq. A.1a,b follow only from the premise of a rank 1 matrix $\mathbf{M}(\mu_0)$ with $M_{22}(\mu_0) = M_{33}(\mu_0) = 0$, regardless of the multiplicity p. If $p \geqslant 3$, then, by substituting Eq. A.1a,b into (5.6b), and taking the element of the resulting matrix equation in the first row and the first column, one obtains

$$v_1 W''_{11}(\mu_0) = 0, \tag{A.2}$$

where $v_1 \neq 0$. But this leads to a contradiction since $W''_{11}(\mu_0)$ cannot vanish along with $W_{11}(\mu_0)$ and $W'_{11}(\mu_0)$. □

Proof of Lemma 2. The lemma will be proved by contradiction. Assuming the contrary to be true, then $\mathbf{W}'(\mu_0)$ cannot have more than one nonzero diagonal element. [Consider, for example, that the two diagonal elements $W'_{11}(\mu_0)$ and $W'_{22}(\mu_0)$ do not vanish. Then the falsity of Lemma 2 implies that $M_{11}(\mu_0) = M_{22}(\mu_0) = M_{33}(\mu_0) = 0$. With $W(\mu_0) = \mathbf{0}$, one then has $M_{23}(\mu_0)^2 = M_{13}(\mu_0)^2 = M_{12}(\mu_0)^2 = 0$, so that the eigenvalue μ_0 is superabnormal, rather than abnormal.] Hence there are only two possible cases: (a) $W'_{11}(\mu_0) = W'_{22}(\mu_0) = W'_{33}(\mu_0) = 0$ and (b) there is an index k such that $W'_{kk}(\mu_0) \neq 0$ and $M_{jj}(\mu_0) = W'_{jj}(\mu_0) = 0$ whenever $j \neq k$. We show in the following that both cases lead to a null matrix $\mathbf{W}'(\mu_0)$, contradicting the assumption that μ_0 is an abnormal-α eigenvalue.

Consider first Case (b). The index k cannot be 1 since $M_{22}(\mu_0)$ and $M_{33}(\mu_0)$ cannot both vanish according to Lemma 1. Furthermore, since the adjoint matrix $\mathbf{W}(\mu_0)$ is null, if a diagonal element $M_{jj}(\mu_0)$ of $\mathbf{M}(\mu_0)$ vanishes then the jth row and column of $\mathbf{M}(\mu_0)$ are null vectors. Hence in Case (b), $\mathbf{M}(\mu_0)$ has only one nonzero element $M_{kk}(\mu_0)$, and k is either 2 or 3. Then the kth row of Eq. (5.6a) implies that the kth row of $\mathbf{W}'(\mu_0)$ is a null vector. Hence the only nonzero diagonal element $W'_{kk}(\mu_0)$ of $\mathbf{W}'(\mu_0)$ must also vanish, so that μ_0 is of the type abnormal-β rather than abnormal-α.

Now consider Case (a). The matrix $\mathbf{W}'(\mu_0)$ is singular since its three columns are all orthogonal to the single independent vector of the rank one matrix $\mathbf{M}(\mu_0)$, as implied by Eq. (5.6a). Hence $Det[\mathbf{W}'(\mu_0)] = 2W'_{12}(\mu_0)W'_{13}(\mu_0)W'_{13}(\mu_0) = 0$. Then the upper-right triangular region of the matrix has only two nonzero off-diagonal elements, and they cannot both vanish since $\mathbf{W}'(\mu_0)$ is not null. Hence $\mathbf{W}'(\mu_0)$ has the following forms, respectively, for the three cases $W'_{12}(\mu_0) = 0, W'_{23}(\mu_0) = 0$ and $W'_{13}(\mu_0) = 0$:

$$\mathbf{W}'(\mu_0) = \begin{bmatrix} 0 & 0 & W'_{13} \\ 0 & 0 & W'_{23} \\ W'_{13} & W'_{23} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & W'_{12} & 0 \\ W'_{12} & 0 & W'_{23} \\ 0 & W'_{23} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & W'_{12} & W'_{13} \\ W'_{12} & 0 & 0 \\ W'_{13} & 0 & 0 \end{bmatrix}, \tag{A.3a,b,c}$$

In each case, the matrix has one and only one nonvanishing row and column, which will be referred to as the ith. Then Eq. (5.6a) requires that the ith row and column of $\mathbf{M}(\mu_0)$ must be null, so that, for $i = 1,2$, and 3, respectively, this symmetric matrix has the following forms:

$$\mathbf{M}(\mu_0) = \begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{12} & M_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} M_{11} & 0 & M_{13} \\ 0 & 0 & 0 \\ M_{13} & 0 & M_{33} \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_{22} & M_{23} \\ 0 & M_{23} & M_{33} \end{bmatrix} \tag{A.4a,b,c}$$

and the two off-diagonal elements of the ith row of Eq. (5.6b) give

$$M'_{33}W'_{13} = M'_{33}W'_{23} = 0, \quad M'_{22}W'_{12} = M'_{22}W'_{23} = 0, \quad M'_{11}W'_{12} = M'_{11}W'_{13} = 0. \tag{A.5a,b,c}$$

For $i = 1$, $M'_{33}(\mu_0)$ cannot vanish along with $M_{33}(\mu_0)$ since $M_{33}(\mu)$ is quartic in μ, whereas W'_{13} and W'_{23} cannot both vanish unless $\mathbf{W}'(\mu_0)$ is null. Hence the two equalities in Eq. (A.5a) imply that $\mathbf{W}'(\mu_0)$ is null. Then the eigenvalue is abnormal-β, contrary to the assumption of this lemma. A similar argument applies to the case $i = 2$ but not to $i = 3$ because $M_{11}(\mu)$ is a quartic function of μ, not quadratic. But if W'_{12} and W'_{13} do not both vanish, then the two equalities in Eq. (A.5c) imply that M'_{11} must vanish along with M_{11}, M_{12} and M_{13}. Then

$$W'_{12} = M_{23}M'_{13} - M_{33}W'_{12}, \quad W'_{13} = M_{23}M'_{12} - M_{22}W'_{13}, \quad (W'_{12})^2 = -M_{33}\boldsymbol{\rho}_1^{\mathrm{T}}\mathbf{M}'(\mu_0)\mathbf{W}'(\mu_0)\boldsymbol{\rho}_1,$$
$$(W'_{13})^2 = -M_{22}\boldsymbol{\rho}_1^{\mathrm{T}}\mathbf{M}'(\mu_0)\mathbf{W}'(\mu_0)\boldsymbol{\rho}_1.$$

However, with $M_{11} = M_{12} = M_{13} = 0$, Eq. (5.6b) yields

$$\boldsymbol{\rho}_1^{\mathrm{T}}\mathbf{M}'(\mu_0)\mathbf{W}'(\mu_0) = -(1/2)\boldsymbol{\rho}_1^{\mathrm{T}}\mathbf{M}(\mu_0)\mathbf{W}''(\mu_0) = \mathbf{0}.$$

Therefore, $(W'_{12})^2 = (W'_{13})^2 = 0$ so that $\mathbf{W}'(\mu_0) = \mathbf{0}$. This again results in a contradiction to the assumption of an abnormal-α eigenvalue. □

Appendix B

(1) **Normal eigenvalue** $p = 1, 2, 3, 4$
Eigenvectors and eigensolutions. Eq. (4.14a,b)
Pseudometrics. Eq. (4.18a–d)

(2) **Abnormal eigenvalue of multiplicity 2**
Eigenvectors. $\mathbf{M}(\mu_0)$ has a nonzero element at the kth diagonal position. Two zeroth-order eigenvectors are $\mathbf{J}(\mu_0)\boldsymbol{\eta}_1(\mu_0)$ and $\mathbf{J}(\mu_0)\boldsymbol{\eta}_2(\mu_0)$, where $\boldsymbol{\eta}_1(\mu)$ and $\boldsymbol{\eta}_2(\mu)$ are given by Eqs. (3.10a), (3.10b) or (3.10c) depending on the value of k.
Pseudometric. Eq. (5.4a).

(3) **Abnormal-$\boldsymbol{\alpha}$ eigenvalue**, $p \geqslant 3$.
The eigenspace is separated into two orthogonal subspaces
First subspace. One zeroth-order eigenvector given by one of the expressions of Eq. (5.10a), depending on the subcase.
Second subspace. Eq. (5.10b) gives one zeroth-order and one first-order eigenvector if $p = 3$, and an additional eigenvector of the second order if $p = 4$.
Pseudometrics. Eqs. (5.11a) and (5.11b), respectively, for $p = 3$ and 4.

(5) **Abnormal-$\boldsymbol{\beta}$ eigenvalue**, $p \geqslant 3$.
The eigenspace is separated into two orthogonal subspaces
First subspace. One zeroth-order and one first-order eigenvector given respectively by Eqs. (5.12a) and (5.13a) if $M_{22}(\mu_0) \neq 0$; otherwise replace $\{0, -M_{33}, M_{22}\}$ by $\{0, M_{33}, -M_{23}\}^{\mathrm{T}}$ in (5.12a) and (5.13a). The pseudometric is given by Eq. (5.16b) with $k = 2$ or 3, depending on whether $M_{22}(\mu_0)$ does not or does vanish.
Second subspace. One zeroth-order eigenvector given by Eq. (5.14a), an additional eigenvector of the first-order given by (5.15a) in the case $p = 4$. The pseudometric is given by Eq. (5.16c) if $p = 3$ and (5.16d) if $p = 4$.

(6) **Superabnormal eigenvalue** $p \geqslant 3$
Eigenvectors. Three zeroth-order eigenvectors are given by the three column of $\mathbf{J}(\mu_0)$ For $p = 4$, Eq. (5.15a) gives a first-order eigenvector.
Pseudometrics. Eqs. (5.17) and (5.19) for $p = 3$, 4, respectively.

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